Ph.D. Dissertation

Interest Rate Derivatives
- Valuation and Applications

by

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1 Thesis Introduction

This Ph.D. thesis has been written during my studies at the Aarhus School of Business. It consists of four self-contained essays on valuation of interest rate derivatives. In particular derivatives related to management of interest rate risk are considered. Financial support to my Ph.D. studies has been provided by ScanRate Financial Systems, a software company developing a commercial software package, RIO, for fixed income and mortgage backed security analysis. This connection is particularly reflected in the first two essays that deal with aspects of the Danish mortgage backed security (MBS) market. The last two essays concern the valuation and exercise of Bermudan swaptions. Their relation to the first two essays might not be obvious at first sight, but many fixed rate mortgage backed securities can be seen as complicated Bermudan swaption structures.

I would like to express my gratitude to my employer ScanRate Financial Systems for giving me the opportunity to write this thesis. In particular thanks to Svend Jakobsen, who has been working as an informal thesis advisor, and to Bo Wase Pedersen who has patiently accepted delays in my work due to my studies. Furthermore, thanks to Nicki S. Rasmussen, Søren Willemann, Malene S. Jensen, Bjarne Nørgaard, and the rest of the ScanRate crew for help, insights and good discussions. Thanks to the faculty, staff, and fellow Ph.D. students at the Department of Finance, it has been a memorable time. Special thanks are due to my advisor Tom Engsted. I also owe thanks to Torben G. Andersen and Ph.D. students at Kellogg Graduate School of Business, Northwestern University for their kind hospitality and for an educational experience during my stay in 2001. Finally, I am indebted to my wife Helle for her unconditional support.

2 English Summaries

2.1 Essay I

“Valuation of Path-Dependent Interest Rate Derivatives in a Finite Difference Setup” was written as a part of an update of existing valuation models in the RIO system. The numerical technique allows the prepayment models used in the valuation of MBSs to include path-dependent explaining variables in the prepayment function, which is important in order to capture the observed prepayment behavior of Danish mortgagors. We study and implement a finite difference version of the augmented state variable approach proposed by Hull & White (1993) that allows valuation of path-dependent securities. We apply the method to a class of path-dependent interest rate derivatives and consider several examples including mortgage backed securities and collateralized mortgage obligations. The efficiency of the method is assessed in a comparative study with Monte Carlo simulation and we find it to be faster for a similar accuracy.

2.2 Essay II

“Mortgage Choice - The Danish Case” is an extended version of an earlier paper written with Svend Jakobsen. Starting from a detailed analysis of a mortgage product recently introduced to the Danish mortgage market and a comparison with more traditional Danish mortgage products, we analyze the mortgage choice facing Danish borrowers. We argue that Adjustable-Rate Mortgages (ARM) with life time caps will combine the most attractive features from straight ARMs and callable Fixed-Rate Mortgages (FRM). Furthermore, we find the delivery option embedded in Danish mortgages to be an important feature, which protects households from the risk of insolvency, by facilitating a closer match between assets and liabilities in household portfolios.

2.3 Essay III

”Efficient Control Variates and Strategies for Bermudan Swaptions in a Libor Market Model” concerns the problem of valuing Bermudan swaptions in a Libor market model. In particular we consider various efficiency improvement techniques for a Monte Carlo based valuation method. We suggest a simplification of the Andersen (2000) exercise strategy and find it to be much more efficient. Furthermore, we test a range of control variates for Bermudan swaptions using a sampling technique for American options proposed in Rasmussen (2002). Application of these efficiency improvements in the Primal-Dual simulation algorithm of Andersen & Broadie (2001), improves both upper and lower bounds for the price estimates. For the Primal-Dual simulation algorithm we examine the variance-bias trade-off between the numbers of outer and inner paths. Here we find that the bias decreases at a rate that is approximately square root two larger than the rate with which the variance decreases. Finally, we demonstrate that stochastic volatility increases the expected losses from following the most simple exercise strategy in Andersen (2000).

2.4 Essay IV

"On the suboptimality of single-factor exercise strategies for Bermudan swaptions", deals with the cost of using recalibrated single-factor models to determine the exercise strategy for Bermudan swaptions in a multi-factor world. We demonstrate that single-factor exercise strategies applied in a multi-factor world only give rise to economically insignificant losses. Furthermore, we find that the conditional model risk as defined in Longstaff, Santa-Clara & Schwartz (2001), is statistically insignificant given the number of observations. Additional tests using the Primal-Dual algorithm of Andersen & Broadie (2001) indicate that losses found in Longstaff et al. (2001) cannot as claimed be ascribed to the number of factors. Finally, we find that for valuation of Bermudan swaptions with long exercise periods, the simple approach proposed in Andersen (2000) is outperformed by the Least Square Monte Carlo method of Longstaff & Schwartz (2001) and, surprisingly, also by the exercise strategies from the single-factor models.

3 Danish Summaries

3.1 Essay I

"Prisfastsættelse af stiafhængige renteafledte derivater i en endelig differens metode" blev skrevet som en del af en opdatering af en eksisterende prisfastsættelsesmodel i analysesystemet RIO. Den anvendte teknik tillader de modeller for konverteringsrater, der anvendes i prisfastsættelsen af konverterbare obligationer, at anvende stiafhængige forklarende variable, hvilket har vist sig at være vigtigt for at kunne forklare den observerede konverteringsadfærd hos danske låntagere.

Vi gennemgår og implementerer en metode introduceret af Hull & White (1993) til prisfastsættelse af stiafhængige aktiver. Vi anvender metoden til renteafledte aktiver, og illustrerer dens anvendelse på blandt andet konverterbare obligationer. Desuden undersøger vi metodens anvendelighed i et komparativt studie med Monte Carlo simulation, og finder at den er hurtigere givet samme nøjagtighed.

3.2 Essay II

"Valg af realkreditlån – det danske tilfælde" er en udvidet udgave af en tidligere artikel skrevet sammen med Svend Jakobsen. Med udgangspunkt i en detaljeret analyse af et nyt realkreditprodukt på det danske marked og en sammenligning med traditionelle lånetyper, analyseres valget af realkreditlån i Danmark. Vi argumenterer for, at variabelt forrentede lån med indbyggede renteloft svarende til amortisationsperioden vil være attraktive for mange af de låntagere, der i dag vælger fast forrentede og almindelige rentetilpasningslån. Desuden argumenterer vi for, at den option, som låntager har på at indfri lånet til kursværdien ved at opkøbe obligationer i markedet, er særdeles vigtig, idet den beskytter låntagerne mod teknisk insolvens, ved at sikre en tættere sammenhæng mellem aktiver og passiver.

3.3 Essay III

"Efficiente kontrolvariate og strategier for Bermuda swaptioner i en Libor-marked model". Dette studie omhandler prisfastsættelsen af Bermuda swaptioner i en Libor-marked model. Specielt undersøger vi forskellige teknikker til at øge effektiviteten i en Monte Carlo baseret prisfastsættelsesmetode. Vi forslår en simplificering af en "exercise"-strategi anvendt i Andersen (2000), og demonstrerer at den er mere efficient. Desuden undersøges en mængde kontrolvariate for Bermuda swaptioner ved hjælp af en sampling teknik foreslået i Rasmussen (2002). Anvendelsen af disse forbedringer i en primal-dual simulationsalgoritme af Andersen & Broadie (2001) forbedrer både øvre og nedre grænser for prisestimaterne. For primal-dual simulationsalgoritmen undersøger vi en varians-bias afvejning mellem antallet af indre og ydre stier. Her finder vi, at bias aftager med en rate, der er ca. kvadratrod to større end den rate, hvormed variansen aftager. Endelig demonstreres det, at stokastisk volatilitet øger de forventede tab ved at følge den simpleste af de strategier, der er foreslået i Andersen (2000).

3.4 Essay IV

"Om suboptimaliteten af en-faktor indfrielsesstrategier for Bermuda swaptioner", omhandler de forventede tab fra anvendelsen rekalibrerede en-faktor modeller til at bestemme indfrielsesstrategien for Bermuda swaptioner i en fler-faktor verden. Vi viser, at en-faktor indfrielsesstrategier anvendt i en fler-faktor verden kun giver anledning til økonomisk insignifikante tab. Desuden viser vi, at den "betingede model risiko" defineret i Longstaff et al. (2001) er statistisk insignifikant givet antallet af observationer. Yderligere test med primal-dual algoritmen af Andersen & Broadie (2001) indikerer at tabene rapporteret i Longstaff et al. (2001) ikke som hævdet kan tilskrives antallet af faktorer. Endelig finder vi at ved prisfastsættelsen af Bermuda swaptioner med lange indfrielsesperioder, bliver den simple metode foreslået i Andersen (2000) overgået af Least-square Monte Carlo metoden af Longstaff & Schwartz (2001), men mere overraskende også af indfrielsesstrategierne fra en-faktor modellerne.

References

Andersen, L. (2000), 'A simple approach to the pricing of bermudan swaptions in the multifactor LIBOR market model', *Journal of Computational Finance* **3**(2), 5–32.

Andersen, L. & Broadie, M. (2001), 'A primal-dual simulation algorithm for pricing multi-dimensional american options', *Working Paper: General Re Securities and Columbia University* .

Hull, J. & White, A. (1993), 'Efficient procedures for valuing european and american path-dependent options', *The Journal of Derivatives* **1**(1), 21–31.

Longstaff, F. A., Santa-Clara, P. & Schwartz, E. S. (2001), 'Throwing away a billion dollars: The cost of suboptimal exercise strategies in the swaptions market', *Journal of Financial Economics* **62**(62), 39–66.

Longstaff, F. A. & Schwartz, E. S. (2001), 'Valuing american options by simulation: A simple least-squares approach', *The Review of Financial Studies* **14**(1), 113–147.

Rasmussen, N. S. (2002), 'Efficient control variates for monte-carlo valuation of american options', *Working Paper, The Aarhus School of Business* .

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Abstract

In this paper we study and implement a finite difference version of the augmented state variable approach proposed by Hull & White (1993) that allows for path-dependent securities. We apply the method to a class of path-dependent interest rate derivatives and consider several examples including mortgage backed securities and collateralized mortgage obligations. The efficiency of the method is assessed in a comparative study with Monte Carlo simulation and we find it to be faster for a similar accuracy.

JEL Codes: G13, G12, C19

Keywords: Path-dependent Options; Finite Difference; Mortgage Backed Securities

1 Introduction

In Hull & White (1993) a method to price path-dependent securities in trees is demonstrated to be an efficient way of handling particular path-dependent securities. The main idea is to augment the state space with additional state variables to represent movements in the past. In Wilmott, Dewynne & Howison (1993) the same technique is applied but in a more general finite differences framework to value exotic options like look-back and Asian options.

In this paper we first summarize the method for interest rate derivatives in a finite difference setup. The method allows us to handle most common features in fixed income products including particular types of path-dependencies as well as American features.

Secondly we apply the technique to other types of path-dependent securities, and we illustrate that the valuation of collateralized mortgage obligations under rational prepayments can be done in a single backward run, as opposed to the two-step procedure proposed in McConnell & Singh (1994) that employs both finite difference and Monte Carlo techniques.

The numerical results presented in Hull & White (1993) indicate that the method is faster and just as accurate as Monte Carlo simulation and that the method is not particularly sensitive to the density of the discretized augmented state space. However, our numerical results show that there are in fact large differences in the density of the augmented state space needed in order for the method to converge, but it is still at least as

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fast as standard Monte Carlo for similar accuracy. The examples we consider are a mortgage backed security (MBS) with a path-dependent prepayment function, collateralized mortgage obligations (CMO) such as the Interest Only (IO), the Principal Only (PO) and Sequential Pay tranches, and, finally, a capped amortizing Adjustable Rate Mortgage (ARM) with a coupon that is settled as an average of historical interest rates.

In section 2 we go through the model framework. Section 3 describes the numerical implementation while section 4 contains applications of the method. Finally, we make our conclusion.

2 The Model Setup

The following exposition is based primarily on Wilmott et al. (1993), and the main difference is that we derive the fundamental partial differential equation in an interest rate model, whereas Wilmott et al. (1993) work in a Black-Scholes world.

2.1 Interest Rate Dynamics

We work in a one-factor term structure setup, with models for the instantaneous short rate r_t that can be represented by the following SDE,

$$dr_t = \mu(r_t, t)dt + \sigma(r_t, t)dW_t,$$

where μ and σ denote drift- and volatility functions that satisfy the usual conditions. W_t is a one-dimensional Wiener-process. This setup covers many of the most commonly used single factor models, but the technique is also applicable to multi-factor models.

Let V denote the value of an interest rate contingent claim, that is dependent on the history of the short rate. Assume that this dependency can be summarized in a z-dimensional state-vector $A \in \mathbb{R}^z$, in the following way

$$A_t = \int_0^t f(r_s, s)ds.$$

To keep notation simple we assume that $z = 1$. However, it is possible to have $z > 1$. With these specifications the value $V(t, r_t, A_t)$ of the claim is Markov with respect to the information generated by the triple (t, r_t, A_t). In other words, we assume that the value of the path-dependent security is given by the real valued function $V(t, r_t, A_t)$ defined on $\mathbb{R}_+ \times \mathcal{D}(r_t) \times \mathcal{D}(A_t)$. Here $\mathcal{D}(\cdot)$ denotes the domain for a given variable. This domain will in general depend on the specific term structure model and the definition of the state-vector. Before continuing notice that

$$dA_t = f(r_t, t)dt,$$

which means that A_t is a state variable of finite variation, and does not add further noise to the system. In particular this means that we do not need to worry about additional risk premia.

2.2 The Partial Differential Equation

A standard arbitrage argument leads to the fundamental partial differential equation for the security (the derivation can be found in appendix A.1 for completeness).

$$\begin{aligned} r_t V(t, r_t, A_t) &= \frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma(r_t, t)^2 \frac{\partial^2 V}{\partial r_t^2} \\ &\quad + (\mu(r_t, t) - \lambda(r_t, t)\sigma(r_t, t)) \frac{\partial V}{\partial r_t} + f(r_t, t)\frac{\partial V}{\partial A_t}. \end{aligned} \tag{1}$$

Here $\lambda(r_t, t)$ denotes the market price of interest rate risk. A terminal condition must be specified in order to determine a single solution to the problem, so let this be given by

$$V(T, r_T, A_T) = h(T, r_T, A_T).$$

With the appropriate boundary conditions, these equations will define the value function and must in general be solved by numerical methods. Observe that the last term in equation (1) is due to the state variable and will be zero for path-independent securities, leaving the usual term-structure equation.

2.3 Discrete Sampling

When the state-variable is updated at discrete time points, the term $\frac{\partial V}{\partial A_t} f(r_t, t)$ in the PDE found above will disappear, as $dA_t = f(r_t, t)\, dt = 0$ between sampling dates. The simplification facilitates the solution compared to the continuous sampling as discrete updates of the state-variable introduce a type of jump condition. Note that in the case of continuous sampling greater care should be taken when implementing this method, but we will not get into the details here, but refer the reader to Forsyth, Vetzal & Zvan (2000) for a rigorous treatment of the numerical aspects.

Let Φ denote the time points where the state variable is updated. By definition discretely sampled state variables remain constant between the sampling dates, and on a sampling date they should be updated through a so-called *update* rule

$$A_{t_i} = U\left(t_i, r_{t_i}, A_{t_{i-1}}\right).$$

A no arbitrage argument (Wilmott et al. (1993)) will show that a corresponding jump condition will be

$$V\left(t_i^-, r_{t_i}, A_{t_{i-1}}\right) = V\left(t_i^+, r_{t_i}, U\left(t_i, A_{t_{i-1}}, r_{t_i}\right)\right), \quad i \in \Phi. \tag{2}$$

In order to provide some intuition for the jump conditions due to discrete sampling of the state variable, consider the following example. Assume we know the current value of the state variable, and that time approaches the next sampling time. The uncertainty regarding the new value of the state variable will diminish and immediately before the fixing time we will know the new value. As the realization of the price process should be continuous when no payments are made to either side of the contract, the values immediately before and after the update should be equal.

It is worth noting that a clever choice of state variable and update rule is important for optimal use of this method. As we shall see later it is sometimes possible to exploit particular properties in a given security or the update rule to reduce the dimensionality of the solution function.

2.4 Discrete Dividends

If the security pays discrete coupons an arbitrage argument leads to jump conditions. Let Ψ denote the set of dates at which the security pays the coupons $D_i\left(t_i, r_{t_i}, A_{t_i}\right)$. Following standard notation let t_i^- and t_i^+ denote the time immediately before and after the $i'th$ payment is made, respectively. This means that the $i'th$ jump condition due to coupons is

$$V(t_i^-, r_{t_i}, A_{t_i}) = V\left(t_i^+, r_{t_i}, A_{t_i}\right) + D_i\left(t_i, r_{t_i}, A_{t_i}\right), \quad i \in \Psi. \tag{3}$$

2.5 Amortization of Principal

Another feature we must be able to incorporate is the amortization of the remaining principal P_t. If t_i is the time where Z_{t_i} units of the principal are repaid, we have

$$V\left(t_i^-, r_{t_i}, P_{t_i^-}, A_{t_i}\right) = V\left(t_i^+, r_{t_i}, P_{t_i^-} - Z_{t_i}, A_{t_i}\right) + Z_{t_i}.$$

If the amortization scheme depends on the interest rate movements it will induce a special kind of path-dependency, but in most cases these value functions have a *similarity solution* without this path-dependency. As demonstrated below, securities where the amortization Z_{t_i} is linear in the remaining principal, support this similarity reduction.

If the amortization schedule Z_{t_i} is defined as a fraction $\theta(t, r_t, A_{t_i})$ of remaining principal, i.e. $Z_{t_i} = \theta(t_i, r_{t_i}, A_{t_i}) \cdot P_{t_i^-}$, then we have the following jump condition

$$V\left(t_i^-, r_{t_i}, P_{t_i^-}, A_{t_i}\right) = V\left(t_i^+, r_{t_i}, (1-\theta(t_i, r_{t_i}, A_{t_i})) \cdot P_{t_i^-}, A_{t_i}\right) + \theta(t_i, r_{t_i}, A_{t_i}) \cdot P_{t_i^-}.$$

For fixed income securities that are homogeneous of first degree in the remaining principal P_t[1], we can apply the similarity reduction

$$V\left(t_i^-, r_{t_i}, P_{t_i^-}, A_{t_i}\right) = (1-\theta(t_i, r_{t_i}, A_{t_i})) \cdot V\left(t_i^+, r_{t_i}, P_{t_i^-}, A_{t_i}\right) + \theta(r_{t_i}, t_i, A_{t_i}) \cdot P_{t_i^-}$$

which implies

$$V\left(t_i^-, r_{t_i}, 1, A_{t_i}\right) = (1-\theta(t_i, r_{t_i}, A_{t_i})) \cdot V\left(t_i^+, r_{t_i}, 1, A_{t_i}\right) + \theta(t_i, r_{t_i}, A_{t_i}) \cdot 1 \qquad (4)$$

This facilitates the solution as we shall find a function of one variable less. We just need to incorporate a version of this jump condition whenever principal is redeemed. Basically, we always measure the value in terms of 100% remaining principal of the security.

3 The Numerical Solution

3.1 Transformation of the PDE

We apply a standard transformation of the interest rate state space (see e.g. Duffie (1996), Stanton & Wallace (1999) or James & Webber (2000)). Define,

$$x(r) = \frac{1}{1+\pi r}, \ \pi > 0,$$

with inverse

$$r(x) = \frac{1-x}{\pi x}, \ \pi > 0.$$

There are mainly two reasons that we want to transform the state space for the spot rate. First, the transformation of the PDE (1) allows us to work with the solution on a bounded space. Secondly, it enables us to increase the number of points in the most relevant part of the state space using the constant π.

Let $u(x,t) = V(r(x), t)$. We now transform the PDE (1) into an PDE in u defined on the bounded state space 0 to 1.

$$\begin{aligned} \frac{\partial V(r,t)}{\partial r} &= \frac{\partial u(x,t)}{\partial x}\frac{\partial x}{\partial r} = u_x \frac{-\pi}{(1+\pi r)^2} = -\pi x^2 u_x, \\ \frac{\partial^2 V(r,t)}{\partial r^2} &= \pi^2 x^4 \frac{\partial^2 u(x,t)}{\partial x^2} + 2\pi^2 x^3 \frac{\partial u(x,t)}{\partial x}. \end{aligned}$$

Substituting into (1) we obtain the following PDE for u in x, t where subscripts are short hand notation for partial derivative

$$\begin{aligned} 0 &= \frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma(r_t,t)^2 \frac{\partial^2 V}{\partial r_t^2} + \tilde{\mu}(r_t,t)\frac{\partial V}{\partial r_t} - V_t r \\ &= u_t + \tfrac{1}{2}\sigma(r(x),t)^2\left(\pi^2 x^4 u_{xx} + 2\pi^2 x^3 u_x\right) + \tilde{\mu}(r(x),t)\left(-\pi x^2 u_x\right) - ur(x) \\ &= u_t + \tfrac{1}{2}\sigma(r(x),t)^2 \pi^2 x^4 u_{xx} + \pi x^2\left(\sigma(r(x),t)^2 \pi x - \tilde{\mu}(r(x),t)\right) u_x - ur(x) \\ &= u_t(x,t) + \beta(x,t) u_{xx}(x,t) + \alpha(x,t) u_x(x,t) - r(x) u(x,t), \end{aligned} \qquad (5)$$

[1] Conditions for similarity reductions must also be satisfied on the boundary as well as by the terminal function.

with terminal condition

$$u(T, x_T, A_T) = h(T, r(x_T), A_T).$$

3.1.1 Boundary conditions.

In general we need to specify boundary conditions if we are using implicit schemes to solve parabolic PDE's. However, as described in Vetzal (1998) for interest rate models with mean reversion and constant standard deviation, the PDE above behaves more like a hyperbolic PDE due to the size of the convection term, even though it is formally parabolic. Hence, it will not only be unnecessary to use boundary conditions, it will actually be most efficient to avoid specifying them.

Unfortunately not many interesting models have constant standard deviation, so we might need to use something else. However, another boundary condition arises naturally, as also noted by Vetzal (1998), by the fact that the $-r(x)\,u$ term causes exponential decay, thereby driving u and its derivatives to zero. This means that in these cases appropriate boundary conditions could be e.g. $u = 0$, $u_x = 0$ or $u_{xx} = 0$ on the upper boundary in r space (lower boundary in x space).

3.2 The Finite Difference Schemes

The PDE in (1) will in general have to be solved numerically, and in this section we describe the finite difference solution used.

Crank-Nicolson and implicit schemes are unconditionally stable, allowing us to match any cash flow, sampling, or decision date. Furthermore, as the Crank-Nicolson scheme is second-order accurate in time, we are able to take much larger steps in the time direction. However, if the terminal condition is not differentiable in the state-variable, the conditions for the Crank-Nicolson scheme are violated, which often causes oscillations in the solution. This can often be avoided by using the pure implicit scheme for the first couple of steps, or by smoothening the payoff function (see e.g. Tavella & Randall (2000)).

Therefore, we will use what is sometimes referred to as the "delta" method, which is basically a convex combination of pure explicit and implicit schemes, with the Crank-Nicolson scheme as the special case with equal weight. This implementation facilitates shifts between different finite difference schemes, by changing the weight ω.

On the boundary we use inside approximations that are second order in space, when applying the implied boundary conditions. We refer to Appendix A.2 for further details.

3.3 Implementing an Augmented State-Variable

To fix some notation let $V^n_{s,k}$, denote the value of the security at time t_n, when the short rate is r_s, and where k denotes level of the state variable. We denote the discretization of the augmented state variable by $\mathcal{A} = \{A_0, ..., A_K\}$. At all sampling times, where the augmented state variable is updated using the update scheme, the value must satisfy the jump condition in (2)

$$V\left(t_j^-, r_{t_j}, A_{j-1}\right) = V\left(t_j^+, r_{t_j}, U\left(t_j, r_{t_j}, A_{j-1}\right)\right) = V\left(t_j^+, r_{t_j}, A_j\right).$$

However, the update function U does not necessarily take values in $\mathcal{A}$, so we will not know the exact value of $V\left(t_j^+, r_{t_j}, A_j\right)$. The basic idea in this method is to approximate it by interpolating the future values at known levels of A.

With a view to this interpolation, define the mapping function $k^*(A) : \mathbb{R} \to \{0, ..., K\}$ by

$$A_{k^*} \leq A < A_{k^*+1}.$$

That is, the mapping picks the index of the highest level of the state variable that is still less than or equal to the value A, assuming that the discretization of the state space has been done such that this is a well-defined mapping. Notice that if V is non-linear in the

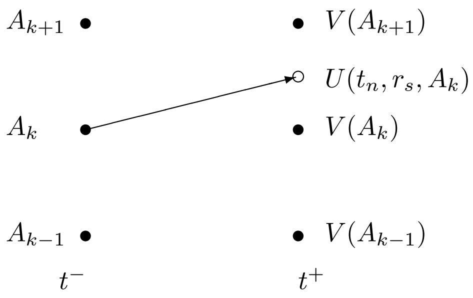

Figure 1: Illustration of the interpolation in the augmented state space.

state variable, we get a biased estimate using linear interpolation. E.g. if V is a convex function of A, then the estimate is too high.

We use either linear- or polynomial interpolation of order 2[2]. Algorithms can be found in Press, Flannery, Teukolsky & Vetterling (1989), and written in a pseudo notation we get

$$V_{s,k}^{n-} = int\left(A_k, \{A_{k^*-1}, A_{k^*}, A_{k^*+1}\}, \left\{V_{s,k^*-1}^{n+}, V_{s,k^*}^{n+}, V_{s,k^*+1}^{n+}\right\}\right).$$

It is possible to make the number of levels of the augmented state space time and state-dependent in order to minimize calculation time, as there is no need to consider levels of the state variable that are not feasible. In situations where the state variable is monotonincally increasing or decreasing, a simple example could be to use the current value as either upper or lower bound of the augmented state space.

4 Applications

The technique can be applied to a wide range of path-dependent securities. The essential part is to make a clever choice of state variables and update rules. As a complicated example Dewynne & Wilmott (n.d.) show how to value a trend based option like a "Five-times-up-and-out" using this approach.

In the following numerical analysis we use the Cox-Ingersoll-Ross model,

$$\mu(r_t, t) = \kappa(\mu - r_t), \quad \sigma(r_t, t) = \sigma\sqrt{r_t}, \quad \lambda(r_t, t) = \lambda^{CIR}\sqrt{r_t}/\sigma$$

with parameters as given in table 1.

κ	μ	σ	λ^{CIR}
0.3	0.08	0.12	0

Table 1: Parameters in the CIR model

4.1 Mortgage Backed Securities

A standard mortgage backed security(MBS) is a fixed rate mortgage with an embedded option that allows the borrower to repay the remaining principal at par at any time during the life of the mortgage. This means that when refinancing rates fall, borrowers prepay their loans by taking up new loans at the prevailing market rate. Any reasonable pricing model for MBS's is designed to incorporate what is known as the burnout effect,

[2] Other interpolation schemes such as cubic splines and rational interpolation have been tested without improvements.

namely that borrowers most inclined to prepay leave the mortgage pool, causing future prepayment rates to decline *ceteris paribus*.

This heterogeneity among borrowers can be modelled in basically two ways, which we will denote explicit and implicit burnout. Implicit modelling of burn out consists of summarizing the historical interest and prepayment behavior in state variables which enter directly into the prepayment function. This is also termed a path-dependent prepayment function. Early contributions in this direction were made by Schwartz & Torous (1989) and Richard & Roll (1989). Examples of explicit modelling of burnout can be found in Jakobsen (1992) and Stanton (1995). By regarding a bond in a large and heterogeneous mortgage pool as a portfolio of homogeneous sub pools, each having a path-independent prepayment function, they demonstrate that changes in the mixture of borrowers will induce a burnout pattern very similar to that of the implicit models.

When it comes to valuing MBS, Monte Carlo simulation has by some been considered superior to backward induction techniques as Monte Carlo simulation allows the prepayment model to combine the two approaches, but as shown here so do recombining lattice methods. On the other hand, Monte Carlo simulation has the serious flaw, namely that it is unable, or at least unsuitable to handle MBSs under rational prepayment behavior. Especially the fact that American or Bermudan option pricing is very hard to do by Monte Carlo simulation, means that we cannot use this approach to compute the optimal prepayment strategy. Furthermore, as mentioned earlier, the finite difference approach facilitates the task of valuing options on MBS's or CMOs as we just use backward induction.

As mentioned above we need to define the state variable and the update rule in order to make use of the method. One variable that has been applied in many prepayment models in various forms is a so called pool factor B_j, that measures the current remaining principal relative to the originally scheduled. If θ_j denotes the conditional prepayment rate, i.e. the fraction of the remaining borrowers that prepay at time t_j, we have that

$$B_j = \Pi_{i=1}^{j} (1 - \theta_i), \quad B_0 = 1.$$

The update rule U is given by

$$B_j = U(t, r_t, B_{j-1}) = B_{j-1} \cdot (1 - \theta_j),$$

Assume that the conditional prepayment rate (CPR) $\theta_j = f\left(t_j, r_{t_j}, B_{j-1}\right)$ is a function f of some explaining variables, one of them being the pool factor, making the prepayment model path-dependent. This means that at a term of notice[3], where the borrowers have to decide whether to prepay or not, we apply the jump condition

$$V(t^-, r_t, B_{j-1}) = \theta_j \cdot 1 + (1 - \theta_j) \cdot V\left(t^+, r_t, U(t, r_t, B_{j-1})\right), \tag{6}$$

measured in terms of principal at time t^-.

4.1.1 MBS: An example

As an example we consider the pricing of a 20-year annuity bond, with a fixed 8% coupon and quarterly payments, where the borrowers' behavior is described by the very simplified but path-dependent prepayment function for the conditional prepayment rate,

$$\theta_j\left(r_{t_j}, B_{j-1}\right) = \min\left((1 + 30 \cdot B_{j-1}) \cdot \left(Coupon - \left(r_{t_j} + 1\%\right)\right)^+, 100\%\right).$$

[3] Almost all mortgages have a term of notice, but in these examples we ignore these features, such that prepayment decisions are taken at the term date.

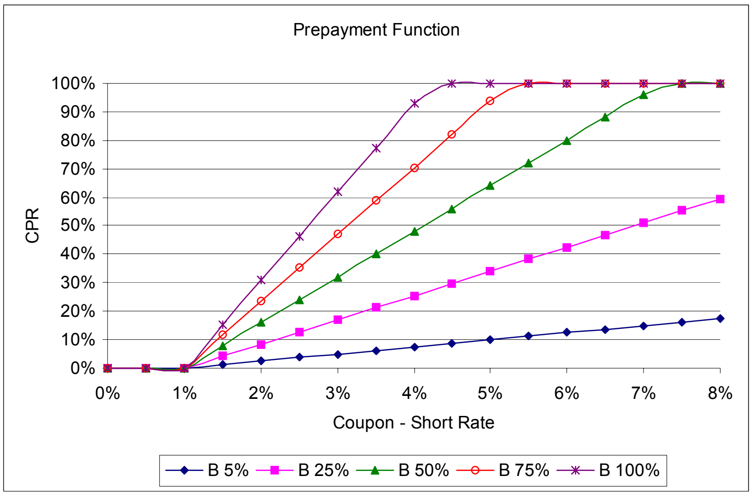

Figure 2: Illustration of the path-dependent prepayment function. CPR denotes the conditional prepayment rate and B the path-dependen burnout factor.

Table 2 illustrates the convergence of the value of the MBS for different values of the short rate as we increase the density of the discrete augmented state space using first linear and then quadratic interpolation. Monte Carlo estimates based on 40.000 paths using antithetic variates as variance reduction (a total of 80.000 paths) are given below. The right column shows the differences between the finite difference solution and the MC estimates measured in basis points. For three out of four levels of the short rate we cannot at a 95% significance level reject the hypotehesis that the MC and PDE values are equal, when the number of states is high enough. However for all practical applications the differences are not significant as they are way inside bid-ask spreads, which are at least 10 bps. These results also confirm the conclusions in Hull & White (1993), namely that the quadratic interpolation seems to improve the method when K is low. The table also shows computation times and it is obvious that the method is quite efficient compared to this particular MC implementation. The interesting thing here is not whether the numbers are exactly equal, as we know that in the limit both methods will give us the correct values. The basic point is that the PDE is fully able to handle this path dependency; there is no need to simulate in this case.

4.2 Collateralized Mortgage Obligations

CMOs are constructed by allocating the payments from the underlying collateral (usually MBS) into different new securities (called tranches). Depending on the redistribution of the payments, these tranches can have characteristics that are indeed very different from those of the collateral. In practice all kinds of CMOs are created to fit investor preferences.

McConnell & Singh (1994) propose a two-step procedure to value CMOs under rational prepayments. The rational exercise of the prepayment option precludes MC as a feasible solution procedure regarding the prepayment decisions, so they find the optimal exercise boundary by finite difference. In the second step they use MC to work forward in time distributing the cash flows using the optimal exercise boundary found in step one as a prepayment function. In relation to the augmented state space approach, McConnell & Singh (1994) claim that it is necessary to include a state variable for each tranche making the approach technically unfeasible. However, if the allocation of the cash flow is based on the remaining debt alone, we do not need more than one state variable per sub pool of borrowers. We do not need a state variable for each tranche.

We now show that some of the most widely used CMO structures can be priced using

Table 2: Convergence of the PDE solution for the Mortgage Backed Security

Linear No. Aug	Comp. Time	PDE Short Rate 2%	4.8%	8%	12%	PDE-MC (bps) 2%	4.8%	8%	12%
3	1	101.484	101.023	96.570	88.949	2	44	46	35
5	1	101.476	100.810	96.317	88.767	1	23	21	16
7	2	101.471	100.728	96.231	88.707	0	15	12	10
9	3	101.469	100.690	96.191	88.679	0	11	8	8
11	3	101.468	100.666	96.169	88.664	0	9	6	6
13	3	101.467	100.651	96.155	88.654	0	7	5	5
15	4	101.466	100.642	96.146	88.648	0	6	4	4
17	4	101.466	100.636	96.140	88.643	0	5	3	4
19	5	101.465	100.630	96.135	88.640	0	5	3	4
21	5	101.465	100.627	96.131	88.638	0	5	2	3
31	8	101.464	100.616	96.122	88.631	0	4	1	3
41	9	101.464	100.612	96.119	88.629	0	3	1	3
61	14	101.464	100.609	96.116	88.627	0	3	1	2
81	18	101.464	100.608	96.115	88.626	0	3	1	2
MC	598	101.47	100.58	96.11	88.60				
Std.Dev		0.00	0.01	0.02	0.02	0.2	1.2	2.1	2.0

Quadratic No. Aug.	Comp. Time	PDE Short Rate 2%	4.8%	8%	12%	PDE-MC (bps) 2%	4.8%	8%	12%
3	1	101.475	100.695	96.169	88.664	1	11	6	6
5	2	101.460	100.512	96.023	88.563	-1	-7	-9	-4
7	2	101.458	100.544	96.064	88.592	-1	-4	-5	-1
9	3	101.460	100.572	96.088	88.609	-1	-1	-2	1
11	4	101.462	100.587	96.101	88.618	-1	1	-1	1
13	5	101.462	100.596	96.108	88.623	0	2	0	2
15	6	101.463	100.603	96.113	88.626	0	2	0	2
17	6	101.463	100.606	96.115	88.627	0	2	1	2
19	7	101.463	100.608	96.117	88.628	0	3	1	3
21	7	101.463	100.609	96.118	88.629	0	3	1	3
31	11	101.463	100.610	96.118	88.629	0	3	1	3
41	14	101.464	100.610	96.117	88.628	0	3	1	2
61	21	101.464	100.608	96.116	88.627	0	3	1	2
81	27	101.464	100.608	96.115	88.626	0	3	1	2
MC	598	101.47	100.58	96.11	88.60				
Std.dev		0.00	0.01	0.02	0.02	0.2	1.2	2.1	2.0

This table illustrates the convergence of the PDE approach using a Linear and Qudratic interpolation scheme. No. Aug denotes the number of spatial grid points in the augmented state-space and Comp. Time is the calculation time in seconds. MC denotes the Monte-Carlo estimates for various levels of initial short rate and Std.Dev. is the standard deviation. PDE-MC is the difference between PDE and the MC estimates measured in basis points. In the PDE implementation a Crank-Nicolson scheme was used with 80 spatial grid points and 24 steps per year.

the augmented state variable approach. There are only a few limitations. For example, we will not be able to calculate measures such as weighted average life (WAL), as the state price distribution in the augmented state space is unknown.

4.2.1 Mortgage Strips

One of the most natural ways to split the total cash flow received from the collateral, is into principal and interest payments. These mortgage strips are also known as Interest Only (IO) and Principal Only (IO). The holder of an IO receives all interest payments from the collateral, while the PO holders receive the scheduled as well as unscheduled repayment on the principal. It is clear that the value of the IO and the PO together should equal that of the collateral, i.e.

$$V^C = V^{IO} + V^{PO}.$$

If we use this fact to rewrite equation (6), it follows that

$$V^C(t^-, r_t, B_{j-1}) = \theta_j \cdot 1 + (1-\theta_j) \cdot V^C\left(t^+, r_t, U\left(t, r_t, B_{j-1}\right)\right)$$

which implies

$$\begin{aligned} V^{IO}(t^-, r_t, B_{j-1}) + V^{PO}(t^-, r_t, B_{j-1}) &= \theta_j \cdot 1 + (1-\theta_j) \cdot V^{IO}\left(t^+, r_t, U\left(t, r_t, B_{j-1}\right)\right) \\ &\quad + (1-\theta_j) \cdot V^{PO}\left(t^+, r_t, U\left(t, r_t, B_{j-1}\right)\right) \end{aligned}$$

The PO receives all repayments, and the IO looses the future interest corresponding to the prepaid principal. Hence, the jump equations due to prepayment will look like

$$\begin{aligned} V^{IO}(t^-, r_t, B_{j-1}) &= (1-\theta_j) \cdot V^{IO}\left(t^+, r_t, U\left(t, r_t, B_{j-1}\right)\right), \\ V^{PO}(t^-, r_t, B_{j-1}) &= \theta_j \cdot 1 + (1-\theta_j) \cdot V^{PO}\left(t^+, r_t, U\left(t, r_t, B_{j-1}\right)\right). \end{aligned}$$

From these it is clear that both these tranches are mildly path-dependent if the prepayment function is path-independent. Hence, the mortgage strips can be evaluated in exactly the same way as the collateral.

4.2.2 Sequential Pay Tranches

As mentioned in section (4.1) there are several reasonable measures for the historical interest rate and prepayment behavior, but the pool factor definition chosen above has the additional advantage that it can also be used to value CMO structures, where we can not apply the similarity reduction. The sequential pay tranches are examples of such structures, as the value of the tranches are *not* linear in remaining debt.

An example Consider two tranches T_1 and T_2 on a collateral of 100 units of the MBS from before. Tranche 1 receives the first W_1 percent of the collateral, and when all

CMO	Nominal	Coupon
Collateral	100	C
Tranche A	$W_1 \cdot 100$	C_1
Tranche B	$W_2 \cdot 100$	C_2

Table 3: Example of Sequential Pay CMO

principal in tranche T_1 has been redeemed, tranche T_2 starts receiving principal. Both tranches receive interest on the remaining principal. Notice that if $C_1 > C_2$ there will be an interest deficit after the first installment on the principal, and an interest excess if $C_1 < C_2$. In these cases issuers often add a residual class - a so called Z-bond, but we will not go into these details. The number of CMO constructions is almost infinite and only the inventiveness seems to set the limit.

To keep things simple let us assume that tranche T_1 receives the first 60% of the principal and that tranche T_2 gets the rest, but that they both pay the same interest as the collateral, i.e. $W_1 = 60\%$, $W_2 = 40\%$, and $C = C_1 = C_2 = 8\%$.

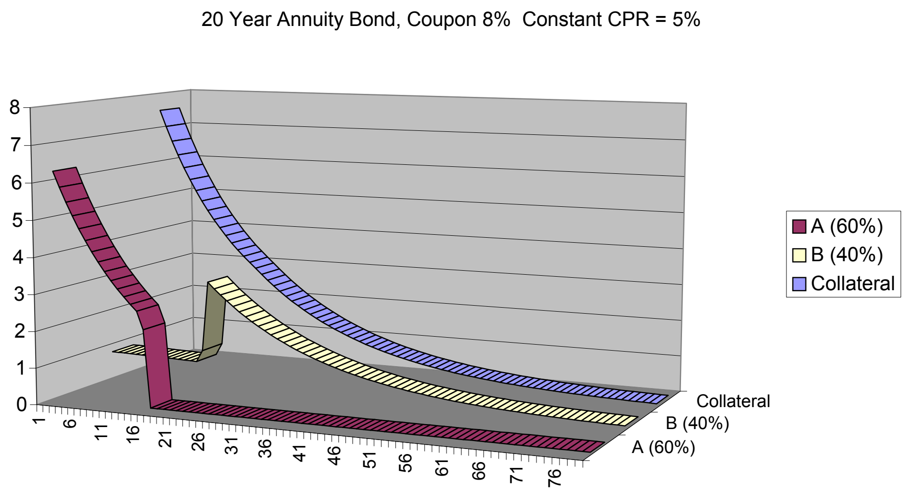

Figure 3: Illustration of cash flows for the tranches, given the cashflow from the collateral.

Notice that this construction has no similarity reduction as the amortization is *not* linear in the principal. Here the use of the augmented state variable is crucial, even if the prepayment function is path-independent. By using the pool factor defined above as state variable, we will be able to decide how much the individual tranches should receive at a given time for a given spot rate. To find the nominal value of the remaining debt we multiply the pool factor with the scheduled remaining principal in case of no prepayments. Denote the scheduled remaining principal in case of no prepayments by $\hat{P}_j$ and the actual remaining principal after the $j'th$ payment P_j, both measured in percent of initial principal. Then by definition of the pool factor,

$$P_j = B_j \cdot \hat{P}_j.$$

Given the nominal value we can allocate the cash flow to the tranches in accordance with the definition as we do when we go forward during the MC simulation. At time j we let Z_j denote the total repayment, TZ_j the total repayment since time 0, I_j^i and Z_j^i denote the interest and repayment for the i'th tranche, and $\bar{I}$ is the number of trances.

$$\begin{aligned}
Z_j &= P_{j-1} - P_j, \\
TZ_j &= 1 - P_j, \\
P_j^i &= \left(\left(\sum_{m=1}^{i} W_m - TZ_j \right) \wedge W_i \right)^+, \quad i = 1, .., \bar{I} \\
Z_j^i &= \left(Z_j - \sum_{m=1}^{i-1} Z_j^m \right) \wedge P_{j-1}^i, \quad i, ..., \bar{I} \\
I_j^i &= P_{j-1}^i \cdot C_i, \quad i = 1, ..., \bar{I}.
\end{aligned}$$

Given these expressions we can now state the following jump condition for the value of tranche i at time j

$$V^i\left(t^-, r_t, B_{j-1}\right) = V^i\left(t^-, r_t, B_j\right) + Z_j^i + I_j^i, \quad i = 1, ..., \bar{I}.$$

Numerical Results for Sequential Tranches As for the numerical results for the sequential tranches reported in Tables 4 and 5, there are at least three things worth mentioning. First, the differences between the MC results and the PDE approach are small when the number of state levels K is high enough. Secondly, however, as opposed to the conclusions in Hull & White (1993) and the results for the collateral, the tranches are quite sensitive with regard to the number of state levels. We need much more than 6 levels in order to obtain reasonable results. Thirdly, we also see that for small values of K the quadratic interpolation scheme performs worse than the linear scheme. The two latter points are not that surprising though, as the value function is not smooth in the state variable.

4.3 Average Rate Capped Amortizing ARM

We examine a security traded in the Danish mortgage market named BoligX. The construction of the security is quite non-standard for several reasons. The BoligX loan is a 5-year adjustable rate mortgage ARM that can be issued with or without an embedded 5-year cap. Usually a cap on an ARM is paid for separately, but the BoligX loan is a genuine pass-through in the sense that payments from the borrowers are paid directly to the mortgage holders, and the cap with strike κ is paid for through a premium rate ρ.

There are quarterly payments which are settled in pairs twice a year. The size of the payments are based on the borrower having an adjustable rate annuity mortgage with m payments, typically 80 or 120 corresponding to 20- or 30-year. The coupon on the underlying mortgage is reset twice a year as a day arithmetic average of the 6-month Cibor rate over a prespecified 10 days fixing period.

This means that at the $n'th$ fixing time, the next two payments are equal to the payment received from an annuity with $m - 2n$ periods and a coupon rate that is C_n. On top of the average will be a coupon premium to pay for the cap. Due to this construction of the security there will be repayment on the principal, and this repayment increases/decreases as interest rates decrease/increase.

Let N denote the number of fixing periods and $\left\{s_n^1, .., s_n^{10}\right\}$ the set of dates in the n'th fixing period. Furthermore, let t_n^1 and t_n^2 denote payment times for the payments settled at time s_n^{10}. Hence, the n'th coupon rate will be given as

$$C_n = \min\left(A_n + \rho, \kappa\right),$$

where $A_n = \frac{1}{10}\sum_{i=1}^{10} r\left(s_n^i\right)$. The size of the payments settled in period n can then be found from the standard annuity formula

$$Y_n = P_n \frac{C_n}{1 - \left(1 + C_n\right)^{-(m-2(n-1))}}, \quad n = 1, ..., \frac{m}{2}$$

where $m - 2\left(n - 1\right)$ is the number of remaining payments and P_n the remaining principal outstanding at fixing time n.

In order to model the settlement of the coupon rate as an average of previous interest rates, we let the state variable A be the discretely sampled average of the short rate . The update rule in the case of a discretely sampled average as a state variable, can be written as

$$A\left(s_n^i\right) = U\left(s_n^i, r\left(s_n^i\right), A\left(s_n^{i-1}\right)\right) = \frac{1}{i} r\left(s_n^i\right) + \frac{i-1}{i} A\left(s_n^{i-1}\right).$$

Another non-standard feature of the BoligX loan is that the sampling takes place before the actual accruement period. But as the payments are known at the fixing time

Table 4: Convergence of the PDE solution for Tranche A

Linear		PDE				PDE-MC			
No.	Comp.	Short Rate				(bps)			
Aug.	Time	2%	4.8%	8%	12%	2%	4.8%	8%	12%
3	3	61.326	69.465	66.295	58.456	47	924	848	494
5	3	60.931	63.081	60.962	55.451	7	285	315	193
7	4	60.885	61.382	59.248	54.469	2	115	143	95
9	5	60.872	60.744	58.501	54.032	1	51	68	51
11	6	60.862	60.346	58.062	53.776	0	12	25	26
13	6	60.866	60.424	57.981	53.662	1	19	16	14
15	8	60.868	60.479	58.019	53.649	1	25	20	13
17	8	60.867	60.453	58.016	53.649	1	22	20	13
19	10	60.865	60.376	57.965	53.631	0	15	15	11
21	10	60.862	60.285	57.897	53.603	0	5	8	8
31	15	60.861	60.267	57.853	53.565	0	4	4	5
41	19	60.861	60.257	57.835	53.553	0	3	2	3
61	29	60.861	60.251	57.821	53.543	0	2	0	2
81	38	60.861	60.248	57.815	53.540	0	2	0	2
MC	574	60.86	60.23	57.82	53.52				
Std.Dev		0.00	0.01	0.01	0.01	0.0	0.5	1.2	1.2

Quadratic		PDE				PDE-MC			
No.	Comp.	Short Rate				(bps)			
Aug.	Time	2%	4.8%	8%	12%	2%	4.8%	8%	12%
3	2	59.862	31.239	21.175	23.957	-100	-2899	-3664	-2956
5	4	59.415	43.296	48.234	49.676	-145	-1693	-958	-384
7	5	60.885	61.382	59.248	54.469	2	115	143	95
9	7	60.872	60.744	58.501	54.032	1	51	68	51
11	8	60.862	60.346	58.062	53.776	0	12	25	26
13	10	60.866	60.424	57.981	53.662	1	19	16	14
15	11	60.868	60.479	58.019	53.649	1	25	20	13
17	12	60.867	60.453	58.016	53.649	1	22	20	13
19	13	60.865	60.376	57.965	53.631	0	15	15	11
21	15	60.862	60.285	57.897	53.603	0	5	8	8
31	21	60.861	60.267	57.853	53.565	0	4	4	5
41	28	60.861	60.257	57.835	53.553	0	3	2	3
61	42	60.861	60.251	57.821	53.543	0	2	0	2
81	54	60.861	60.248	57.815	53.540	0	2	0	2
MC	574	60.86	60.23	57.82	53.52				
Std.dev		0.00	0.01	0.01	0.01	0.0	0.5	1.2	1.2

This table illustrates the convergence of the PDE approach using a Linear and Qudratic interpolation scheme. No. Aug denotes the number of spatial grid points in the augmented state-space and Comp. Time is the calculation time in seconds. MC denotes the Monte-Carlo estimates for various levels of initial short rate and Std.Dev. is the standard deviation. PDE-MC is the difference between PDE and the MC estimates measured in basis points. In the PDE implementation a Crank-Nicolson scheme was used with 80 spatial grid points and 24 steps per year.

Table 5: Convergence of the PDE solution for Tranche B

Linear		PDE				PDE-MC			
No.	Comp.	Short Rate				(bps)			
Aug.	Time	2%	4.8%	8%	12%	2%	4.8%	8%	12%
3	3	40.115	30.847	29.542	29.967	-49	-950	-875	-512
5	3	40.517	37.326	34.971	33.045	-9	-302	-332	-204
7	4	40.566	39.083	36.740	34.067	-4	-127	-155	-102
9	5	40.582	39.755	37.519	34.528	-2	-60	-77	-56
11	6	40.594	40.177	37.980	34.799	-1	-17	-31	-29
13	6	40.591	40.115	38.075	34.924	-2	-24	-22	-16
15	8	40.590	40.071	38.047	34.943	-2	-28	-25	-14
17	8	40.592	40.105	38.057	34.949	-1	-25	-24	-13
19	10	40.595	40.190	38.115	34.971	-1	-16	-18	-11
21	10	40.598	40.286	38.187	35.002	-1	-6	-11	-8
31	15	40.600	40.319	38.244	35.050	-1	-3	-5	-3
41	19	40.601	40.337	38.267	35.065	-1	-1	-3	-2
61	29	40.601	40.348	38.287	35.079	0	0	-1	-1
81	38	40.602	40.354	38.295	35.083	0	0	0	0
MC	574	40.61	40.35	38.29	35.08				
Std.Dev		0.00	0.01	0.01	0.01	0.1	0.8	1.0	0.9

Quadratic		PDE				PDE-MC			
No.	Comp.	Short Rate				(bps)			
Aug.	Time	2%	4.8%	8%	12%	2%	4.8%	8%	12%
3	2	41.588	69.304	74.925	64.667	98	2895	3663	2958
5	4	42.185	59.347	49.621	40.072	158	1900	1133	499
7	5	40.613	39.634	37.205	34.385	1	-72	-109	-70
9	7	40.613	40.111	37.815	34.728	1	-24	-48	-36
11	8	40.616	40.422	38.181	34.934	1	7	-11	-15
13	10	40.607	40.292	38.218	35.019	0	-6	-7	-6
15	11	40.603	40.205	38.153	35.013	0	-15	-14	-7
17	12	40.602	40.208	38.138	35.002	0	-14	-16	-8
19	13	40.603	40.271	38.177	35.012	0	-8	-12	-7
21	15	40.605	40.351	38.237	35.034	0	0	-6	-5
31	21	40.604	40.348	38.263	35.061	0	0	-3	-2
41	28	40.603	40.350	38.276	35.070	0	0	-2	-1
61	42	40.602	40.352	38.289	35.080	0	0	0	0
81	54	40.602	40.355	38.296	35.084	0	0	0	0
MC	574	40.61	40.35	38.29	35.08				
Std.Dev		0.00	0.01	0.01	0.01	0.1	0.8	1.0	0.9

This table illustrates the convergence of the PDE approach using a Linear and Qudratic interpolation scheme. No. Aug denotes the number of spatial grid points in the augmented state-space and Comp. Time is the calculation time in seconds. MC denotes the Monte-Carlo estimates for various levels of initial short rate and Std.Dev. is the standard deviation. PDE-MC is the difference between PDE and the MC estimates measured in basis points. In the PDE implementation a Crank-Nicolson scheme was used with 80 spatial grid points and 24 steps per year.

s_n^{10}, the time s_n^{10} present value of the payments settled is just $Y_n\left(B\left(s_n^{10},t_n^1\right)+B\left(s_{n,}^{10}t_n^2\right)\right)$, where $B(t,T)$ denotes the time t value of a discount bond maturing at time T.

A part of these two payments is amortized principal, and hence we will need to incorporate this into a jump condition. It can easily be shown that the amortization rate θ due to the first two payments of an annuity bond with an initial nominal of P_n, $m-2(n-1)$ payments and coupon C_n is

$$\theta_n = \frac{Y_n \cdot (1+C_n)^{-(m-2(n-1))} \cdot (2+C_n)}{P_n}.$$

We are now ready to state the jump conditions for the sampling dates in period n

$$V(s_n^{j-}, r(s_n^j), A(s_n^{j-1})) = V(s_n^{j+}, r(s_n^j), A(s_n^j)), \quad j = 1, ..., 9$$

At the last sampling date in period n we also add the present value of the two payments and apply the jump condition due to the armortized principal.

$$\begin{aligned} V(s_n^{10-}, r\left(s_n^{10}\right), A\left(s_n^9\right)) &= Y_n \cdot \left(B(s_n^{10}, t_n^1) + B(s_n^{10}, t_n^2)\right) \\ &\quad + (1-\theta_n) \cdot V\left(s_n^{10+}, r\left(s_n^{10}\right), A\left(s_n^{10}\right)\right), \end{aligned}$$

At the very last payment date the investor also receives the remaining principal, while the intermediary issues a new BoligX loan on behalf of the borrower.

Numerical Results BoligX The premium ρ in the example is 20 bps and the cap rate κ is 7.7%. In table 6 we see that only the out the money value is more than two standard deviations away from the MC value. As there are no differences in the performance of the linear or quadratic interpolation scheme when $K \geq 5$, there is no reason to use anything other than linear interpolation.

5 Conclusions

In this paper we have analyzed a numerical method that efficiently allows valuation of a class of path-dependent interest rate derivatives in a finite difference setup. We have focused on mortgage backed security valuation in particular and we show that this method is able to handle both the American feature but also path-dependencies present in MBS's. Furthermore, the method is at least as efficient as standard Monte Carlo techniques for similar precision, even when we consider 20- or 30-year products.

There are of course limitations to the application of this method due to the curse of dimensionality. If the dimension of the augmented state vector is high, we will not only have to make use of a high dimensional interpolation scheme, but the number of points in the discretized augmented state space will increase exponentially with the dimension. For example, suppose we have a mortgage pool that consists of say 4 sub pools or more with different prepayment behavior. The valuation of a sequential pay CMO, would require us to use a 4-dimensional state vector to summarize all possible combinations of remaining debt or equivalently burnout in the sub pools.

At last we mention that this method can also be used to model and access the value of the delivery options embedded in for example Danish mortgage backed bonds. A delivery option gives the borrower the right to buy back her own loan from the mortgage pool at market value. The presence of this option means that we almost never see prepayments below par. In order to model this option we will need to know the market value of the mortgage at each future time and state. That is, we need not only know the values of the loans in individual sub pools but also their relative share of the total mortgage pool.

Table 6: Convergence of the PDE solution for the BoligX bond

Linear		PDE				PDE-MC			
No.	Comp.	Short Rate				(bps)			
Aug.	Time	2%	4.8%	8%	12%	2%	4.8%	8%	12%
3	3	98.637	97.438	94.827	89.093	-23	-37	-37	-17
5	3	98.675	97.501	94.871	89.116	-19	-31	-33	-14
7	4	98.722	97.570	94.917	89.139	-15	-24	-28	-12
9	4	98.758	97.627	94.958	89.161	-11	-18	-24	-10
11	4	98.783	97.627	94.993	89.179	-9	-14	-21	-8
13	6	98.801	97.698	95.022	89.193	-7	-11	-18	-7
15	6	98.815	97.719	95.046	89.205	-5	-9	-15	-5
21	8	98.841	97.761	95.098	89.232	-3	-5	-10	-3
31	11	98.861	97.794	95.147	89.256	-1	-2	-5	0
41	14	98.873	97.813	95.179	89.272	0	0	-2	1
61	20	98.882	97.827	95.204	89.284	1	2	1	3
81	25	98.882	97.828	95.205	89.285	1	2	1	3
101	32	98.882	97.828	95.205	89.285	1	2	1	3
MC	475	98.87	97.81	95.20	89.26				
std.dev		0.008	0.011	0.013	0.011	0.8	1.1	1.3	1.1

Quadratic		PDE				PDE-MC			
No.	Comp.	Short Rate				(bps)			
Aug.	Time	2%	4.8%	8%	12%	2%	4.8%	8%	12%
3	3	99.356	98.948	98.111	96.586	49	114	291	733
5	4	98.675	97.501	94.871	89.116	-19	-31	-33	-14
7	4	98.722	97.570	94.917	89.139	-15	-24	-28	-12
9	5	98.758	97.627	94.958	89.161	-11	-18	-24	-10
11	6	98.783	97.668	94.993	89.179	-9	-14	-21	-8
13	7	98.801	97.698	95.022	89.193	-7	-11	-18	-7
15	8	98.815	97.719	95.046	89.205	-5	-9	-15	-5
21	10	98.841	97.761	95.098	89.232	-3	-5	-10	-3
31	14	98.861	97.794	95.147	89.256	-1	-2	-5	0
41	19	98.873	97.813	95.179	89.272	0	0	-2	1
61	27	98.882	97.827	95.204	89.284	1	2	1	3
81	36	98.882	97.828	95.205	89.285	1	2	1	3
101	44	98.882	97.828	95.205	89.285	1	2	1	3
MC	475	98.87	97.81	95.20	89.26				
std.dev		0.008	0.011	0.013	0.011	0.8	1.1	1.3	1.1

This table illustrates the convergence of the PDE approach using a Linear and Qudratic interpolation scheme. No. Aug denotes the number of spatial grid points in the augmented state-space and Comp. Time is the calculation time in seconds. MC denotes the Monte-Carlo estimates for various levels of initial short rate and Std.Dev. is the standard deviation. PDE-MC is the difference between PDE and the MC estimates measured in basis points. In the PDE implementation a Crank-Nicolson scheme was used with 80 spatial grid points and 24 steps per year.

References

Andersen, L. B. G. (1996), Five Essays on the Pricing of Contingent Claims, PhD thesis, The Aarhus School of Business.

Dewynne, J. & Wilmott, P. (n.d.), Untitled.

Duffie, D. (1996), *Dynamic Asset Pricing*, Princeton University Press.

Forsyth, P., Vetzal, K. & Zvan, R. (2000), 'Convergence of lattice and PDE methods for pricing asian options', *Working Paper, University of Waterloo* .

Hilliard, J., Kau, J., Keenan, D. & Muller III, W. (1995), 'Pricing a class of american and european path dependent securities', *Management Science* **41**(12), 1892–1899.

Hull, J. & White, A. (1993), 'Efficient procedures for valuing european and american path-dependent options', *The Journal of Derivatives* **1**(1), 21–31.

Jakobsen, S. (1992), Prepayment and the Valuation of Mortgage-Backed Bonds, PhD thesis, The Aarhus School of Business.

James, J. & Webber, N. (2000), *Interest Rate Modelling*, John Wiley and Sons, Ltd.

McConnell, J. & Singh, M. (1994), 'Rational prepayments and the valuation of collateralized mortgage obligations', *The Journal of Finance* **XLIX**(3), 891–921.

Press, W., Flannery, B., Teukolsky, S. & Vetterling, W. (1989), *Numerical Recipes in Pascal*, Cambridge University Press.

Richard, S. & Roll, R. (1989), 'Prepayments on fixed-rate mortgage backed securities', *Journal of Portfolio Management* pp. 73–82.

Schwartz, E. & Torous, W. (1989), 'Prepayment and the valuation of mortgage-backed securities', *The Journal of Finance* **XLIV**(2), 375–392.

Stanton, R. (1995), 'Rational prepayment and the valuation of mortgage-backed securities', *Review of Financial Studies* **8**, 677–708.

Stanton, R. & Wallace, N. (1999), 'Anatomy of and ARM: The interest-rate risk of adjustable-rate mortgages', *Journal of Real Estate Finance and Economics* **19**(1), 49–67.

Tavella, D. & Randall, C. (2000), *Pricing Financial Instruments - The Finite Difference Method*, John Wiley and Sons, Inc.

Vetzal, K. (1998), 'An improved finite difference approach to fitting the initial term structure', *The Journal of Fixed Income* pp. 62–81.

Wilmott, P., Dewynne, J. & Howison, S. (1993), *Option Pricing: Mathematical Models and Computation*, Oxford Financial Press.

A Appendix

A.1 The Derivation of the Fundamental PDE

If we assume that the value function $V(t, r_t, A_t)$ satisfies the usual regularities we can apply Itô's lemma to find the dynamics for the value of the claim

$$
\begin{aligned}
dV(t, r_t, A_t) &= \frac{\partial V}{\partial r_t} dr_t + \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial A_t} dA_t + \tfrac{1}{2} \frac{\partial^2 V}{\partial r_t^2} d\langle r_t \rangle \\
&= \left(\mu(r_t, t) \frac{\partial V}{\partial r_t} + \frac{\partial V}{\partial t} + \tfrac{1}{2} \sigma(r_t, t)^2 \frac{\partial^2 V}{\partial r_t^2} + f(r_t, t) \frac{\partial V}{\partial A_t} \right) dt \\
&\quad + \sigma(r_t, t) \frac{\partial V}{\partial r_t} dW_t.
\end{aligned}
$$

Let now $X_t = F(r_t, t)$ denote the price of another security depending on the short rate e.g. a zero coupon bond of maturity T, governed by the following SDE

$$
\begin{aligned}
dX_t &= \frac{\partial F}{\partial r_t} dr_t + \frac{\partial F}{\partial t} dt + \tfrac{1}{2} \frac{\partial^2 F}{\partial r_t^2} d\langle r_t \rangle \\
&= \frac{\partial F}{\partial r_t} dr_t + \frac{\partial F}{\partial t} dt + \tfrac{1}{2} \sigma(r_t, t)^2 \frac{\partial^2 F}{\partial r_t^2} dt
\end{aligned}
$$

If we sell α_t units of X_t and for each unit of V, we see that the value of the portfolio changes as

$$
\begin{aligned}
d(V_t - \alpha_t X_t) &= dV_t - \alpha_t dX_t \\
&= \left(\frac{\partial V}{\partial r_t} dr_t + \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial A_t} dA_t + \tfrac{1}{2} \sigma(r_t, t)^2 \frac{\partial^2 V}{\partial r_t^2} dt \right) \\
&\quad - \alpha_t \left(\frac{\partial F}{\partial r_t} dr_t + \frac{\partial F}{\partial t} dt + \tfrac{1}{2} \sigma(r_t, t)^2 \frac{\partial^2 F}{\partial r_t^2} dt \right)
\end{aligned}
$$

By choosing α_t such that

$$
\left(\frac{\partial V}{\partial r_t} - \alpha_t \frac{\partial F}{\partial r_t} \right) dr_t = 0 \Leftrightarrow \alpha_t = \frac{\partial V / \partial r_t}{\partial F / \partial r_t},
$$

the change in value of the portfolio is deterministic. Hence the drift should be equal to the short rate,

$$
\begin{aligned}
d(V_t - \alpha_t X_t) &= d\left(V_t - \frac{\partial V / \partial r_t}{\partial F / \partial r_t} X_t \right) = \left(V_t - \frac{\partial V / \partial r_t}{\partial F / \partial r_t} X_t \right) r_t dt \\
&\Leftrightarrow \\
&\left(\frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial A_t} dA_t + \tfrac{1}{2} \sigma(r_t, t)^2 \frac{\partial^2 V}{\partial r_t^2} dt \right) \\
&- \frac{\partial V / \partial r_t}{\partial F / \partial r_t} \left(\frac{\partial F}{\partial t} + \tfrac{1}{2} \sigma(r_t, t)^2 \frac{\partial^2 F}{\partial r_t^2} \right) dt = \left(V_t - \frac{\partial V / \partial r_t}{\partial F / \partial r_t} X_t \right) r_t dt \\
&\Leftrightarrow \\
&\frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial A_t} dA_t + \tfrac{1}{2} \sigma(r_t, t)^2 \frac{\partial^2 V}{\partial r_t^2} dt - V_t r_t dt \\
&= \frac{\partial V / \partial r_t}{\partial F / \partial r_t} \left(\frac{\partial F}{\partial t} + \tfrac{1}{2} \sigma(r_t, t)^2 \frac{\partial^2 F}{\partial r_t^2} - X_t r_t \right) dt \\
&\Leftrightarrow \\
&\frac{\left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma(r_t, t)^2 \frac{\partial^2 V}{\partial r_t^2} - V_t r + \frac{\partial V}{\partial A_t} f(r_t, t) \right)}{\partial V / \partial r_t} dt \\
&= \frac{\left(\frac{\partial F}{\partial t} + \frac{1}{2} \sigma(r_t, t)^2 \frac{\partial^2 F}{\partial r_t^2} - X_t r_t \right)}{\partial F / \partial r_t} dt
\end{aligned}
$$

As the securities were arbitrarily chosen, the equation cannot depend on them, hence leaving the right hand side equal to some function $g(r_t, t)$ depending only on time and the short rate. A standard trick is to write this as a function of the drift and the volatility for some function $\lambda(r_t, t)$ which is denoted market price of risk. Define $\lambda(r_t, t)$ such that $g(r_t, t) = -(\mu(r_t, t) - \lambda(r_t, t)\sigma(r_t.t))$, leading to

$$\begin{aligned}
g(r_t, t) &= \frac{\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma(r_t, t)^2 \frac{\partial^2 V}{\partial r_t^2} - V_t r + \frac{\partial V}{\partial A_t} f(r_t, t)\right)}{\partial V / \partial r_t} \\
&\Leftrightarrow \\
0 &= \frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma(r_t, t)^2 \frac{\partial^2 V}{\partial r_t^2} - V_t r_t + \frac{\partial V}{\partial A_t} f(r_t, t) - g(r_t, t)\frac{\partial V}{\partial r_t} \\
&\Leftrightarrow \\
r_t V_t &= \frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma(r_t, t)^2 \frac{\partial^2 V}{\partial r_t^2} + \frac{\partial V}{\partial A_t} f(r_t, t) + (\mu(r_t, t) - \lambda(r_t, t)\sigma(r_t, t))\frac{\partial V}{\partial r_t}.
\end{aligned}$$

A.2 The Finite Difference Schemes

We will use the "delta" method in order facilitate shifts between various finite difference schemes, thus letting

$$\begin{aligned}
\frac{\partial}{\partial t} u(x_s, t_n) &= \frac{u_s^{n+1} - u_s^n}{\delta_t^n} + O(\delta_t^n), \\
\frac{\partial}{\partial x} u(x_s, t_n) &= \omega_1 \frac{u_{s+1}^n - u_{s-1}^n}{2\delta_x} + (1 - \omega_1)\frac{u_{s+1}^{n+1} - u_{s-1}^{n+1}}{2\delta_x} + O\left((\delta_x)^2\right), \\
\frac{\partial^2}{\partial^2 x} u(x_s, t_n) &= \omega_1 \frac{u_{s+1}^n - 2u_s^n + u_{s-1}^n}{(\delta_x)^2} + (1 - \omega_1)\frac{u_{s+1}^{n+1} - 2u_s^{n+1} + u_{s-1}^{n+1}}{(\delta_x)^2} + O\left((\delta_x)^2\right).
\end{aligned}$$

Notice, that setting ω_1 equal to 1 corresponds to a pure implicit scheme, 0 to a pure explicit scheme and ω_1 equal to $\frac{1}{2}$ is the Crank-Nicolson scheme. To simplify the notation let

$$\begin{aligned}
\beta(x, t) &= \tfrac{1}{2}\sigma^2\pi^2 x^4, \\
\alpha(x, t) &= \pi x^2\left(\sigma^2 \pi x - \tilde{\mu}\right), \\
\gamma(t) &= \frac{\delta_t^n}{(\delta_x)^2}, \\
\omega_2 &= 1 - \omega_1.
\end{aligned}$$

Substituting the finite difference approximations into the PDE (5) and simplifying, we get the following equation for an inner point (n, s) of the grid

$$\begin{aligned}
&-\omega_1 l_s^n u_{s-1}^n + (1 + \omega_1 \delta_t^n r(x_s) - \omega_1 m_s^n) u_s^n - \omega_1 h_s^n u_{s+1}^n \\
&= \omega_2 l_s^n u_{s-1}^{n+1} + (1 - \omega_2 \delta_t^n r(x_s) + \omega_2 m_s^n) u_s^{n+1} + \omega_2 h_s^n u_{s+1}^{n+1},
\end{aligned}$$

where

$$\begin{aligned}
m_s^n &= -2\gamma\beta, \\
l_s^n &= \gamma\left(\beta - \tfrac{1}{2}\alpha\delta_x\right), \\
h_s^n &= \gamma\left(\beta + \tfrac{1}{2}\alpha\delta_x\right).
\end{aligned}$$

For the two boundary equations we will use implied boundary conditions, but we will not be able to use a central derivative to approximate u_{xx} and u_x. Furthermore, as we do not wish to spoil the second order of the Crank-Nicholson scheme, by using a simple one sided difference approximation, which is only accurate to order $O(\delta_x)$. Instead we will use the following one sided approximations when x is at the boundaries, as they are accurate of order $O\left((\delta_x)^2\right)$. On the upper boundary x_{S+1}

$$\begin{aligned}
\frac{\partial}{\partial x}u\left(x_{S+1},t_n\right) &= \omega_1\frac{u^n_{S-1}-4u^n_S+3u^n_{S+1}}{2\delta_x}+\omega_2\frac{u^{n+1}_{S-1}-4u^{n+1}_S+3u^{n+1}_{S+1}}{2\delta_x}, \\
\frac{\partial}{\partial x\partial x}u\left(x_{S+1},t_n\right) &= \omega_1\frac{u^n_{S-1}-2u^n_S+u^n_{S+1}}{(\delta_x)^2}+\omega_2\frac{u^{n+1}_{S-1}-2u^{n+1}_S+u^{n+1}_{S+1}}{(\delta_x)^2}.
\end{aligned}$$

which lead to

$$\begin{aligned}
& -\omega_1\gamma\left(\beta+\tfrac{1}{2}\alpha\delta_x\right)u^n_{S-1}+\omega_1 2\gamma\left(\alpha\delta_x+\beta\right)u^n_S \\
& +\left(1+\omega_1 r(x_{S+1})\delta^n_t-\omega_1\alpha\tfrac{3}{2}\gamma\delta_x-\omega_1\beta\gamma\right)u^n_{S+1} \\
= \; & \omega_2\gamma\left(\beta+\tfrac{1}{2}\alpha\delta_x\right)u^{n+1}_{S-1}-\omega_2 2\gamma\left(\alpha\delta_x+\beta\right)u^{n+1}_S \\
& +\left(1-\omega_2 r(x_{S+1})\delta^n_t+\omega_2\alpha\tfrac{3}{2}\gamma\delta_x+\omega_2\beta\gamma\right)u^{n+1}_{S+1}.
\end{aligned}$$

Similar approximations on the lower boundary x_0 lead to

$$\begin{aligned}
& \left(1+\omega_1 r(x_0)\delta^n_t+\omega_1\gamma\left(\alpha\tfrac{3}{2}\delta_x-\beta\right)\right)u^n_0 \\
& +\omega_1 2\gamma\left(\beta-\alpha\delta_x\right)u^n_1-\omega_1\gamma\left(\beta-\alpha\tfrac{1}{2}\delta_x\right)u^n_2 \\
= \; & \left(1-\omega_2 r(x_0)\delta^n_t-\omega_2\gamma\left(\alpha\tfrac{3}{2}\delta_x-\beta\right)\right)u^{n+1}_0 \\
& -\omega_2 2\gamma\left(\beta-\alpha\delta_x\right)u^{n+1}_1+\omega_2\gamma\left(\beta-\alpha\tfrac{1}{2}\delta_x\right)u^{n+1}_2.
\end{aligned}$$

These equations can be expressed as follows. Notice, that the one sided, but second order, approximations come with a (very) small price tag, namely that the system of equations that we end up with, is not a truly tri-diagonal system.

$$\begin{bmatrix}
B_0 & C_0 & D_0 & 0 & & & \\
A_1 & B_1 & C_1 & 0 & & & \\
0 & A_2 & B_2 & C_2 & & & \\
 & & & \cdots & & & \\
 & & A_{S-1} & B_{S-1} & C_{S-1} & 0 & \\
 & & 0 & A_S & B_S & C_S & \\
 & & 0 & E_{S+1} & A_{S+1} & B_{S_{+1}} &
\end{bmatrix}
\begin{bmatrix} u^n_0 \\ u^n_1 \\ u^n_2 \\ \vdots \\ u^n_{S-1} \\ u^n_S \\ u^n_{S+1} \end{bmatrix}
= \mathbf{rh}\left(\mathbf{u}^{n+1}\right),$$

$$\mathbf{rh}\left(\mathbf{u}^{n+1}\right)=\begin{bmatrix}
b_0 & c_0 & d_0 & 0 & & & \\
a_1 & b_1 & c_1 & 0 & & & \\
0 & a_2 & b_2 & c_2 & & & \\
 & & & \cdots & & & \\
 & & a_{S-1} & b_{S-1} & c_{S-1} & 0 & \\
 & & 0 & a_S & b_S & c_S & \\
 & & 0 & e_{S+1} & a_{S+1} & b_{S_{+1}} &
\end{bmatrix}
\begin{bmatrix} u^{n+1}_0 \\ u^{n+1}_1 \\ u^{n+1}_2 \\ \vdots \\ u^{n+1}_{S-1} \\ u^{n+1}_S \\ u^{n+1}_{S+1} \end{bmatrix}.$$

However, we will only need two additional row operations in order to obtain a true tri-diagonal system. Then we can use standard routines to solve the system.

A.3 The Monte Carlo Setup

The simulation setup used in this paper is based on the excellent paper on efficient simulation in non-linear one-factor interest rate models by Andersen (1996). We apply the extended version of the second order Milstein discretization scheme and the antithetic variate technique for variance reduction. Admittedly, there are several techniques that could possibly reduce the variance of the Monte Carlo estimates further.

Mortgage Choice - The Danish Case

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Mortgage Choice - The Danish Case*

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Abstract

In this paper we analyze the mortgage choice faced by Danish borrowers. Based on an analysis of the most popular Danish mortgage products, we argue that Adjustable-Rate Mortgages (ARM) with life time caps will combine the most attractive features from straight ARMs and callable Fixed-Rate Mortgages (FRM). Furthermore, we find the delivery option embedded in Danish mortgages to be an important feature, which protects households from the risk of insolvency by facilitating a closer match between assets and liabilities in the household portfolio.

JEL Codes: D10;G11;G12;G21;

Keywords: Mortgage; Choice;Valuation; Delivery Option; Prepayment;

1 Introduction

The Danish market for mortgage backed bonds is more than 200 years old, but the basic principles have remained the same[1]. It is characterized by its relative simplicity and a high degree of efficiency. The so-called mortgage credit institutions fund loans issued to borrowers by selling an equal amount of bonds in the markets. A strict balance principle in the legislation requires the mortgage credit institutions to have a very close match between the payments on the loans and the bonds issued, which basically means that all issues are pass-throughs. Furthermore, independent of the borrowers creditworthiness, the maximum loan-to-value LTV limits are 80% for residential property and 60% for corporate property. In effect, Danish mortgage backed bonds are considered highly-secure investment grade securities with ratings from Moody's ranging from Aaa to Aa2. In fact, in the 200-year history not a single investor has received less than the full payment and not a single mortgage credit institution has gone bankrupt.

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[1] More detailed information can be found in *Mortgage Financing in Denmark* (1999).

Over the years changes in the mortgage legislation as well as tax-laws have also affected the maturity and amortization profiles of the loans issued. The vast majority of the mortgages are still callable annuity bonds with maturities of 30-, 20- or 10-years. However, several loan types have been, and still are, available e.g. Inflation-indexed Fixed-Rate Mortgages (IFRM).

In 1989 an amendment to the existing mortgage law allowed new mortgage credit institutions to be created, and an effect of this has been a significant increase in competition. The close connection between bonds and loan profiles makes bonds from different mortgage credit institutions very close substitutes, which ultimately forces mortgage credit institutions to make costs and markups the primary parameters in a very competitive market. However, an increasingly important parameter has been the development of new mortgage products in order to maintain (and attract) customers. In our opinion the increased competition and the focus on market shares have lead the mortgage credit institutions to develop mortgage products that appear attractive at prevailing market conditions, focusing primarily on low initial payments.

The mortgage choice is the single most important financial decision most households are going to make. As discussed in e.g. Campbell & Cocco (2002), this decision requires considerations that are on the frontier of finance research, including uncertainty in interest rates and inflation, risky labor income, borrowing constraints, and illiquid assets.

In the US mortgage choice literature the mobility of the borrower is a key issue. A consistent finding is that mobile and wealthy borrowers as well as short term housing tenures tend to prefer short-term mortgages (see e.g. Dhillon, Schilling & Sirmans (1987), Brueckner & Follain (1988), Aadu & Sirmans (1995)). A closely related part of the US literature argues that points and coupons are ways to make borrowers reveal private information regarding their mobility (see e.g. Leroy (1996), Stanton & Wallace (1998)).

In a very recent study Campbell & Cocco (2002) implement a life-cycle model with interest rate and income risk. Furthermore, the model includes mobility for non-financial reasons, a variety of mortgage contracts including second mortgages, and the wealth effect of the property value. Their numerical results support the findings that low probability of moving, large houses and high risk aversion, increase the preferences for FRMs. Their main conclusion is that inflation-indexed FRMs are preferred for household risk management. However, this effect is decreasing in inflation uncertainty. Also interestingly, they find that hybrid ARMs with caps and floors are more attractive than both straight ARMs and nominal FRMs.

However, there is one reason in particular as to why we cannot apply the conclusions from the US case in Denmark. Danish mortgage backed bonds have an additional option embedded, often denoted the delivery option (or buy back option). This option allows a mortgagor to cancel his loan by buying back bonds at market value, effectively cashing in the capital gain. This means that no prepayments are observed whenever prices are below par, but most importantly the delivery option alleviates the mobility issue known from the US and hence makes most US mortgage choice studies less applicable in the Danish case.

As we shall see, the delivery option sustains a better match between the mortgagor's assets and liabilities, and therefore we often hear the opposite advice in Denmark: borrowers that are more likely to move should issue FRMs close to par. The reason is that if they are likely to move, they should be willing to pay the higher coupon in a FRM for a short while, in return for protection against increasing interest rates just prior to a house sale.

The literature on mortgage choice in Denmark is rather limited considering the long history and no single study has taken all the elements considered in Campbell & Cocco (2002) into consideration. Recent research in Nielsen & Poulsen (2002*a*) and Nielsen & Poulsen (2002*b*) gives a partial explanation. Focusing on the interest rate risk using an advanced two-factor stochastic programming approach, they are able to support the prepayment behavior historically observed in Denmark, including both prepayments and deliveries. However, they do not consider early redemption for non-financial reasons (death, divorce, job relocation), and therefore favor straight ARMs to FRMs in their implementation, as the delivery option is less worth in a straight ARM due to the low interest rate sensitivity. Finally, they do not consider ARMs with caps.

Indexed linked FRMs have been available in Denmark for decades, but have gradually become less popular. This is interesting considering the conclusions made by Campbell & Cocco (2002), but could well be related to the relative low inflation uncertainty in the Euro area, however we are not aware of any studies of this subject.

In this paper we examine a hybrid ARM, Bolig-X (BLX), recently introduced in the Danish mortgage market and compare it to existing mortgage types. Furthermore, we suggest modifications that in our opinion would make it even more attractive. Even though we do not apply a formal utility based analysis, our conclusion is not far from the one in Campbell & Cocco (2002). Given that mortgagors are risk averse, it must be important to ensure a reasonable match between assets and liabilities in the household portfolio. A main point in this paper is that the Bolig-X loans with cap[2] are surprisingly reminiscent of callable FRMs. Both loan types protect the mortgagor from increases in interest rates and the mortgage payments can be reduced when interest rates decline. The existing Bolig-X loans are, however, too short to provide the mortgagor with a sufficient protection against increases in interest rates, but if the maturity of the loan and hence the cap is increased to e.g. 30 years, then these bonds would from a mortgagor's view appear at least as attractive as traditional callable FRMs.

The paper is organized as follows. We start out by giving a short introduction to delivery options as well as the most common existing mortgage products in Denmark. Then we go through the calculation technique behind the Bolig-X loans, and discuss the valuation technique and their price-yield relationship. With model calculations as benchmark we compare Bolig-X loans with straight ARMs, ARMs with a connected guarantee, as well as callable FRMs. After that we estimate the payments that mortgagors have to pay in order to have a life time cap on the coupon. Finally, we complete the treatment with a discussion. In all calculations it is assumed that mortgagors amortize their debt over 30 years.

[2] If it is not explictly expressed in the text, the term Bolig-X loan will refer to the type with a cap.

2 The Delivery Option

The so-called delivery option is a feature embedded in most Danish mortgages. It allows a mortgagor to buy back his "own" bonds in the market and deliver these bonds to the mortgage credit institution that will cancel his old loan. Basically all mortgages on the Danish mortgage market are pass-through securities, and as a result of this close connection between bonds and loans the delivery option is easily implemented in practice. The delivery option has mostly been discussed in relation to dynamic debt management strategies that use the delivery option to switch to a higher coupon when interest rates have gone up. The purpose of this strategy is to reduce debt and to get higher interest payments which are tax deductable (see e.g. Jakobsen (1992)). Furthermore, the new mortgage provides them with a more valuable option if interest rates are going to decrease again.

However, what many Danish households fail to appreciate is that the delivery option also ensures a much closer match between assets and liabilities. This is particularly important if households are likely to redeem their mortgage for non-financial reasons, as they will have to realize any mismatch between the value of their property and their debt. Hence, the delivery option is more worth to mortgagors which are more likely to move.

3 Loan types

3.1 Callable Fixed Rate Mortgages (FRM)

Callable FRMs still account for the majority of the mortgage market in Denmark. Typically they have maturities of 10, 20, or 30 year and are callable at par. The bond series are open for issues for a period of three years. Loans can be issued as either cash- or bond loans.

Contrary to the US case mortgagors are not directly presented with a menu of coupons and points. If a mortgagor decides to take out a callable FRM e.g. a callable annuity, with a certain coupon and maturity, the mortgage institution sells a corresponding bond on the stock exchange and just transfers the proceeds to the mortgagor. However, given the combination of amortization profile, coupon, maturity and call option the price of this bond will only by chance trade at par. Furthermore, due to tax-reasons the mortgage credit institutions do not issue bonds trading above par, so the proceeds from a bond sale will generally be lower than the initial principal. This means that a higher principal value is required to obtain a given revenue from the sale. This capital loss works basically in the same way as the coupons and points system in the US.

However, a capital loss can be made tax-deductable in Denmark by issuing a so called cash-loan. On a cash-loan the coupon rate is the yield-to-maturity on the day the bond is issued, which is higher than the coupon on the corresponding bond loan (cash loans are only meaningful when prices are below par). This way the capital loss is transferred into tax-deductable interests and therefore capital gains from redeeming cash-loans are liable to taxation.

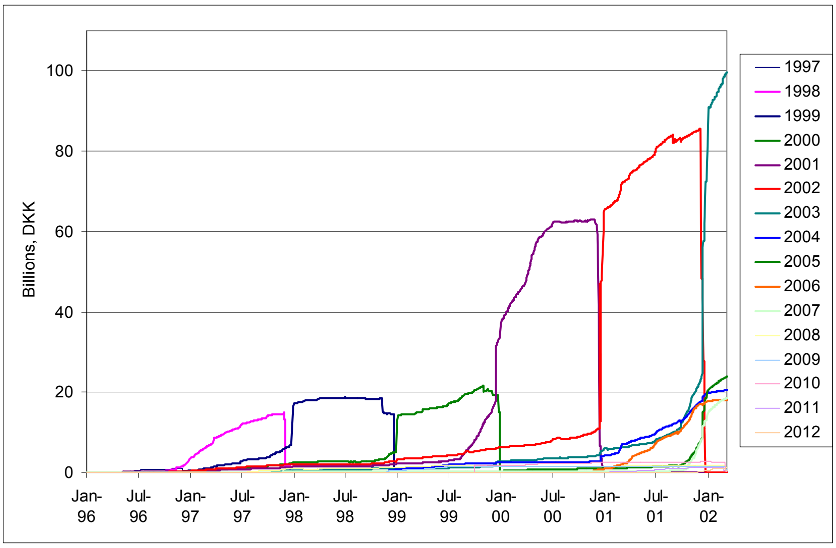

Figure 1: This figure illustrates the nominal outstanding of non-callable bullet bonds used to fund the ARMs in the Danish mortgage market. The figure illustrates how a new 11 year bullet bond is issued when the current 1-year bond matures. The bond series are aggregated based on their maturity indicated in the legend.

3.2 Adjustable Rate Mortgages (ARM)

As interest rates have declined during the last decade Adjustable Rate Mortgages (ARMs) have gained a footing on the Danish mortgage market and now account for a large fraction of new issues. The mortgage credit institution RD's Flexloans (Flexlån) are probably the most well-known ARMs, but similar products are offered by other mortgage credit institutions, including Nykredit, Unikredit, DLR and BRF. Most loans are so called F1-loans, which are funded with 1 year bullet bonds. The advantage of these loans is that the mortgagor refinances his mortgage at the currently low 1 year rate (plus the contribution fee). The disadvantage is that the mortgagor only knows the mortgage payments one year ahead. Figure 1 shows the rapid increase in the nominal amounts of the underlying non-callable bullet bonds sold to finance the ARMs.

These loans are issued as cash loans, but we refer to Tørnes-Hansen (1997) for a thorough introduction.

3.3 Bolig-X mortgages (BXL)

In April 2000 a new adjustable rate product Bolig-X was introduced by Totalkredit. These loans are issued as 5 year bond loans where the coupon is reset twice a year to the 6 Month CIBOR rate[3]. Until now 6 different Bolig-X bond series

[3] CIBOR rates are published daily by the Danish Central Bank (Danmarks Nationalbank) on the basis of quotes of interbank loan rates from currently 8 Danish banks. Rates of 1, 2, 3, 4, 5, 6, 9 and 12 Months loans are published.

have been issued, but in this paper we focus on the original three since these are more liquid; 5.156% CIBOR 122c 2005 (abbreviated BX-05 in this paper) is a straight ARM, where the coupon and hence mortgage rate follows the 6 Month CIBOR rate. In the other two bonds 5.356% CIBOR 122c 2005 (BXL-05) and 2007 (BXL-07) the mortgagors pay CIBOR plus a yield spread of 20 basis points (0.2%) for an embedded cap, which guarantees that the coupon rate cannot exceed 7.7% at any time. An overview of the three bond series is given in table 1.

Name	**Coupon**	**Expiry**	**Price Oct. 1, 2001**	**Amount DKK mil.**
BX-05	5.156%	Jan–01-2005	100.50*	451
BXL-05	5.356%	Jan-01-2005	100.67	5,763
BXL-07	5.356%	Jan-01-2007	100.06	6,702

* Last price Sep 13, 2001

Table 1: Overview Bolig-X loans, October 2001

The Bolig-X loan without cap is a simple way to construct an F1-like mortgage and is very much like the ARMs known from the US mortgage market. At first sight the Bolig-X loan with cap corresponds closely to an F1-loan with an interest rate guarantee as offered by other mortgage credit institutions. However, where the interest rate guaranties offered by the other mortgage credit institutions have failed to take on[4], almost all of Totalkredit's customers have chosen to pay for an embedded cap. This is seen in figure 2.

The success of these loans is likely to be the main reason why Totalkredit has won market shares during the last couple of years.

The Bolig-X bonds have quarterly term dates but the coupon rate is settled twice a year. The coupon rate for the terms October and January is determined as the average of 6 Month CIBOR during the 10 first trade dates in May while the coupon rate for term dates April and July is set during the 10 first trade dates in November. The coupon rate is therefore known approximately 1.5 month before the term period begins, and as such the coupon rate for 1, 2 or 3 terms could be known depending on the trade date.

The mortgagor repays the loan as an annuity with a computational maturity of e.g. 30 years. An example of a possible cash flow is given in table 2.

With a principal of 1 mil. DKK, a coupon rate of 5.36% and 120 terms to maturity we get a quarterly payment of on term date Jan, 02 of 16,801 DKK. April 2002 the coupon rate is set at 4.3% and with 119 terms left and a remaining principal of 996,599 the payment is 14.883 etc. As is evident a lower coupon results in a higher repayment and vice versa.

April/July 2003 and 2004 illustrate periods where CIBOR plus 20 bp is higher than the cap rate and hence the coupon is set at 7.7%. In January 2005 the bond matures and the mortgagor has to repay the remaining principal of 962.489[5] DKK. In order to make this payment the mortgagor will have to issue

[4]To our knowledge no information is published regarding the interest rate guaranties and our assessment is based on conversations with employees at the mortgage institutions.

[5]After some year there could be very different maturities in the BoligX loans as old loans with e.g. 15 years to maturity are to be refinanced in the same bond series as new 30 year mortgages. This will influence the repayment profile and in particular the value of the bonds.

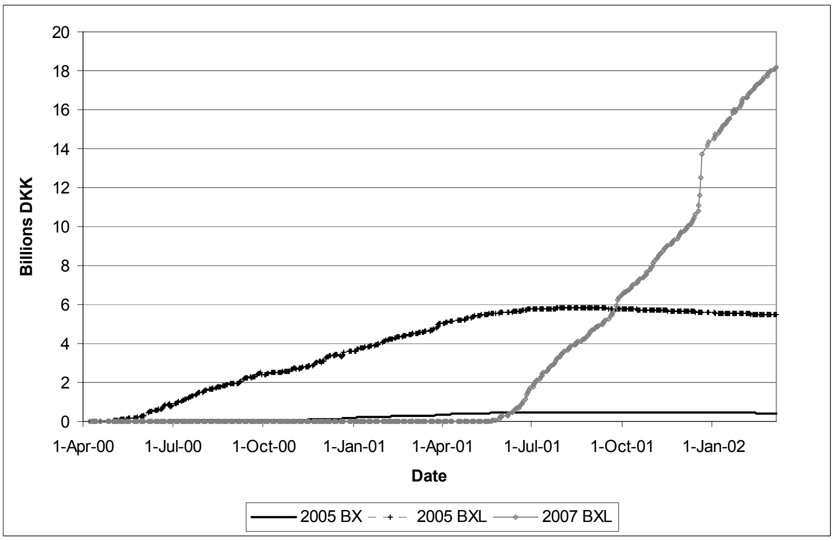

Figure 2: This figure shows the issues in the three bonds series from April 2000 until April 2002. It is obvious that the two series BXL 05 and BLX 07 with the embedded cap are far more popular among mortgagors than the BX 05 with cap.

a new bond at the prevailing level of interest rates.

The Bolig-X loan can be seen as a package consisting of an adjustable rate bond and a sold cap on 6 month CIBOR with strike 7.7%, and with a stochastic notional following the amortized at the same rate as the underlying bond. A cap is a standardized financial contract which for a given principal pays the difference between the current interest rate and a fixed rate. Prices of caps written on CIBOR are quoted on a regular basis by the major banks and are available on Reuters and Bloomberg terminals. There are, however, problems by using this information directly:

- The cap in the Bolig-X bonds is written on a principal that is reduced in line with the mortgagors repayments. The repayments are unknown in advance and vary systematically with the development in the interest rates. Increasing interest rates means that the notional of the cap decreases less and hence makes the cap more valuable.
- The cap is written on a 10 day average of CIBOR, which reduces the volatility and hence the caps value relative to a cap written on CIBOR.
- The cap is written on a 6 month rate but with quarterly terms, the strike rate is 7.7% and not equal to the strike rate on the quoted caps, and finally the underlying coupon rate is determined 1.5 month before the term date in contrast to the quoted caps which are settled at CIBOR at the beginning of each term.

We assess that a minimum and maximun maturity within the individual bond series will be necessary.

Term Date	Coupon	Remaining Principal	Interest	Repayment	Payment
Jan, 02	5.36%	996,599	13,400	3,401	16,801
Apr, 02	4.30%	992,429	10,713	4,169	14,883
Jul, 02	4.30%	988,215	10,669	4,214	14,883
Oct, 02	5.80%	984,950	14,329	3,265	17,594
Jan, 03	5.80%	981,638	14,282	3,312	17,594
Apr, 03	7.70%	979,264	18,897	2,374	21,271
Jul, 03	7.70%	976,844	18,851	2,420	21,271
Oct, 03	6.40%	973,725	15,630	3,119	18,748
Jan, 04	6.40%	970,557	15,580	3,169	18,748
Apr, 04	7.70%	967,998	18,683	2,559	21,242
Jul, 04	7.70%	965,390	18,634	2,608	21,242
Oct, 04	7.20%	962,489	17,377	2,901	20,278
Jan, 05	7.20%	0	17,325	962,489	979,814

Table 2: Possible cashflow stream for a Bolig-X loan 2005 with a principal of 1 mill. DKK and a coupon rate cap of 7.7 pct.

- The cap is out of the money and hence less liquid.

These problems make it more difficult to apply simple models to accurately value the cap. For the calculations in this paper we have chosen to use a term structure model which allows us to take these issues into account.

4 Valuation of the mortgage products

4.1 Interest rate model and numerical implementation

The pricing model used in the following analysis is a one-factor Cox, Ingersoll & Ross (1985) term structure model. The short rate r_t dynamics under the risk neutral measure is given by the diffusion

$$dr_t = \kappa(t)(\mu(t) - r_t)\,dt + \sigma(t)\sqrt{r_t}dW_t^Q.$$

In the examples we have chosen to use constant parameters with a current short rate of 3.7% to facilitate replication of the results. The parameters used here was a mean reversion level $\kappa = 20\%$, volatility parameter $\sigma = 0.06$ and the long term level $\mu = 7\%$. These parameters provide a reasonable fit to the Danish swap curve and ATM caps on October 1th, 2001 even though we tend to overvalue the long caps. The choice of the CIR model was made primarily to have some degree of skew in the implied volatilities of caps.

We apply the implementation of a Crank-Nicholson finite-difference solution described in Svenstrup (2002) to solve the fundamental PDE

$$r_t V(t, r_t, A_t) = \frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial r_t^2} + \mu \frac{\partial V}{\partial r_t} + f(r_t, t)\frac{\partial V}{\partial A_t}, \tag{1}$$

where $V(t, r_t, A_t)$ denotes the time t value of an interest rate dependent claim when the short rate is r_t. A_t denotes an additional state variable used during the fixing periods to capture the path dependency of the running average of the 6 month CIBOR. Notice, as the state variable A_t is updated discretely the last term in the PDE is actually replaced by additional jump conditions.

The implementation takes into account that the coupons are fixed 2-8 months before they are actually paid and hence we only add the present value of the coupons at each fixing date. During the fixing periods we also keep track of the additional state variable representing the accumulated average, but notice that this is not necessary between fixing periods as the state variable is the 6 Month CIBOR on the first fixing day in each period.

The idea with the chosen model is solely to give a quantitative estimate of the effects of the various input parameters. In practice the model applied would have time dependent parameters in order to obtain a closer fit to the observed term structures of interest rates and volatilities.

4.2 A prepayment model for FRMs

The prepayment model used in the valuation of the FRMs is a version of the ScanRate and Reuters DMBS model based on prepayment data for the period July 1997 to July 2001 (see Pedersen (2000)).

The standard references for a Danish prepayment model are Jakobsen (1992) and Jakobsen (1995). Basically all prepayment models used in the Danish market are variations and minor extensions of the required gain prepayment model developed in Jakobsen (1992). This is a hazard-rate based model for the conditional prepayment rate λ. It stipulates that mortgage holders require a certain net present value gain from prepayment in order to make it worth the effort. Furthermore, the model utilizes information of the borrower composition in a bond series, published by the mortgage credit institutions, to create sub-pools of mortgage holders based on the size of their debt etc.

According to the model the fraction of mortgagors in sub-pool i who prepay at time t can be estimated with

$$\lambda_i(t) = \Phi\left(\boldsymbol{\beta} \cdot f_i\left(x_i; \boldsymbol{\alpha}_i\right)\right),$$

where Φ denotes the standard normal distribution, $\boldsymbol{\beta}$, $\boldsymbol{\alpha}_i$ are paramter vectors, f_i a set of basis functions and x_i a set of covariates. The vector of covariates x_i includes the net present vaue gain from prepaying, the time to maturity and a path-dependent burn-out factor. For further details on Danish prepayment models and the valuation of callable FRMs, see Jakobsen & Svenstrup (1999) and Jakobsen & Rasmussen (1999).

In the valuation of the FRMs we use the additional state-variable in equation (1) to model the path-dependency due to the so-called burn-out effect. Notice that contrary to the US case (see e.g. Schwartz & Torous (1989), Richard & Roll (1989)), we do not include mobility related covariates in the prepayment model.

5 Bolig-X model prices

We now consider the valuation results for the Bolig-X loans. Figure 3 illustrates the model prices on October 1th, 2001 for the three Bolig-X bonds as a function of different levels of the yield curve[6]. Initially the yield premium and cap rate of the bonds are adjusted so as to make the bonds trade close to par. For high interest levels the prices decrease for the two bonds with the cap and the effect is most pronounced for the one with the longest maturity BXL-07.

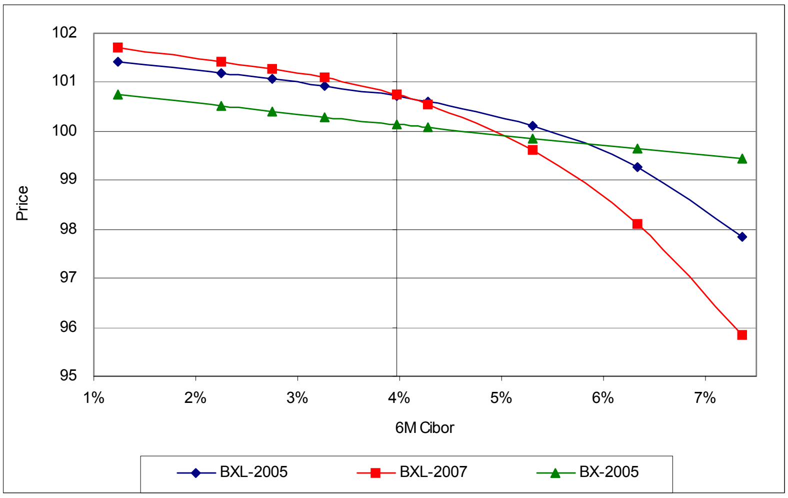

Figure 3: The model prices on October 1th-2001 for Totalkredits Bolig-X bonds at different interest rate levels. The vertical line indicates the initial level of the 6 month CIBOR.

A closer examination shows some interesting effects. On October, 1th the coupon rate for the Jan, 2nd term is fixed at 5.16% for BX-2005. This means that the price is above 100 at low interest rate levels and below for high interest rates. If the interest rates are low the prices of BXL-05/07 with caps are higher than the BX-05 without cap. This is due to a 20 basis points yield premium used to pay for the cap, while the value of the cap is almost zero at low rates. BXL-07 has the yield premium for two years longer than the BXL-05 and hence has a marginally higher price than BXL-05. On the other hand, if the interest rate level is sufficiently high the effect of the cap dominates and the price on BXL-07 is lower than BXL-05. In our calibration of the model to market data on October 1th, 2001, the short rate is 3.7% corresponding to a 6 Month CIBOR on 3.97%. This level of interest rates is marked by the vertical line in the figure. The model gives approximately the same prices for BXL-05 and BXL-07 while the market as shown in table 1 assigns the BXL-05 the highest value.

6 Bolig-X and Adjustable Rate Mortgages

The Bolig-X loan without cap has properties that are very similar to an ARM of the F1 type. As an experiment, let us assume that Totalkredit had chosen to

[6] We have chosen to shift the short rate in the CIR model and maintain the distance between the short rate and its long term level. This will result in almost parallel shifts in the entire yield curve. The first coupon rate January,1th 2002 is fixed at the actual coupon rate of 5.16/5.36%. The subsequent rates vary as determined by the term structure model.

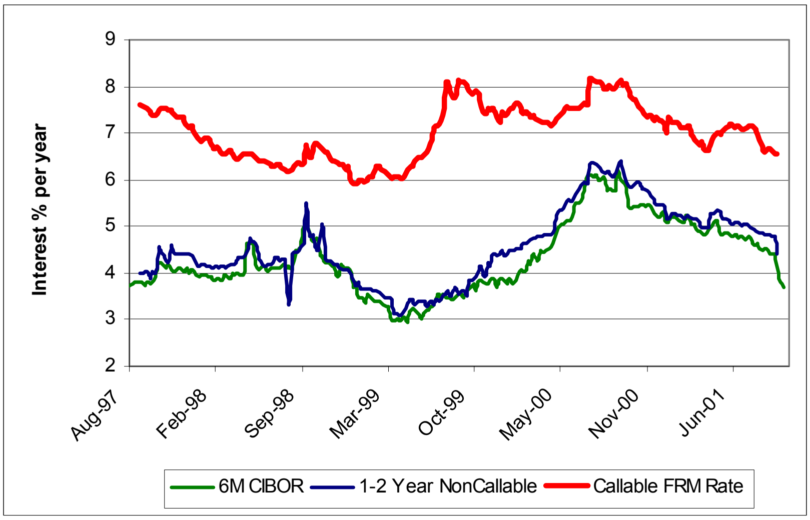

Figure 4: Source: Reuters, The Association of Danish Mortgage Banks

issue BF1 loans without cap where the coupon rate in the beginning of December every year was set equal to the cash loan rate that RD obtains at the auction where they refinance their F1 loans. At the same time assume it is possible for BF1 loans to be sold at price 100. The effect would be that BF1 mortgage holders would have the exact same payment profile before and after tax as the RD mortgage holder with an F1 loan.

On the other hand it will not be certain that BF1 bonds would actually be sold at par. There is a small tax free profit due to appreciation for investors in F1 bonds which are typically sold below par. The tax free profit could result in a smaller payments on F1 loans relative to BF1 loans.

In addition the investors would have to assess the liquidity of the BF1 bond relative to an investment where they roll over F1 bonds. A lower liquidity would depress the price on the BF1 bonds.

However, Totalkredit has chosen not to use the F1 coupon but instead 6 month CIBOR as index. As is evident from figure 4 there is a very high correlation between these two rates. On average the 6 month CIBOR is a bit lower than the F1 rate.

CIBOR is defined as a reference rate for loans to a Prime Bank on unsecured basis, while Danish Mortgage Backed Securities are loans backed by property. When buying Bolig-X bonds the investor should hence make an assessment for the remaining maturity of the loan, whether the credit risk of Danish households will be higher or lower than on a CIBOR based loan. With one year ARMs on the other hand investors have the possibility to adjust prices in case of changes in the credit risk.

To summarize, there are differences between plain Bolig-X loans and Flexloans with yearly adjustment. These differences are solely related to differences in transaction costs, tax issues and credit risk and it would be difficult before hand to assess whether Totalkredit's bonds would be priced higher or lower than traditional straight ARMs.

7 Interest rate caps and guaranties

There should be an obvious need to put a ceiling over the payments on an ARM and among others RD and Nykredit offer 1-5 year interest guaranties where the mortgagor can choose a cap rate of e.g. 7% or 8%, see e.g. Thomsen & Tørnes-Hansen (2000) and Bondorf, Sørensen & Carlsen (2000). These interest rate guaranties are sold as supplementary products to ordinary Danish ARMs.

By selling the interest rate guarantee as a separate product they manage to keep the underlying ARMs simple and liquid and at the same time provide the mortgagors with high flexibility with respect to the design of the guarantee. Despite this, these products have been almost ignored by mortgage holders. In complete contrast the majority of Totalkredit's customers have preferred to pay a yield premium to get a 5 year coupon rate cap.

There are probably several reasons for this. Interest guaranties are sold by a bank attached to the individual mortgage credit institution and until now the guaranties have likely been too expensive relative to the market value of the corresponding option. Furthermore, on early redemption of the mortgage the interest rate guarantee has to be sold back to the bank at a price determinded by the bank, which is probably going to be lower than the market value.

So far the interest rate guarantee has been paid either up-font or through the contribution rate. The first method is not tax-deductable and it is likely that the last method wil be rejected by the tax authorities. Under all circumstances, redemption or transferring of the loan will result in full taxation of any profits on the interest rate guarantee.

In contrast the payment of the guarantee is embedded in the Bolig-X loan as a yield premium on currently 20 basis points. The cap is priced as a part of the bond and hence we expect a relatively sharp pricing. After a severe increase in interest rates the mortgagor could cash in the value of the cap by exercising the delivery option and buy his bonds back in the market. As Bolig-X loans are issued as bond loans any profits are not taxable. The yield premium is, however, fully tax-deductable as well as any transferring of the loan will not release further taxes.

To summarize, we argue that the flexibility which should characterize interest rate guaranties on ARMs is drowned in transaction costs and in the fact that the tax authorities require a close connection between interest rate guarantee and the underlying bond. The Bolig-X construction has from the beginning omitted that flexibility. In return the mortgagor gets the interest rate insurance at the lowest possible price both before and after tax. Last but not least is the marketing. Totalkredit has unambiguously marketed the Bolig-X loan with cap while e.g. RD and Nykredit have hardly marketed their corresponding interest rate guarantee products.

8 Adjustable-Rate Mortgages and callable Fixed-Rate Mortgages

Figure 5 shows the price-yield relationship for the two Bolig-X loans together with a traditional callable fixed-rate annuity 6% 2032 (6-32 denotes coupon and maturity). There are obvious differences. 6-32 is prepayable at price 100 but despite this, the price is getting over 100 because the investors know from experience that many mortgagors are slow to take advantage of their right to prepay. That is investors receive a high coupon for a longer time. Only when all mortgagors decide to prepay the price will be 100. This corresponds to the investor getting the return on his investment reset to the market level.

The coupon rate of the ARM automatically follows changes in the market rates. Hence for investors this corresponds to a situation where all mortgagors are following an optimal prepayment strategy without transaction costs. The ARM with a 6% cap is therefore less worth to investors than the corresponding callable 6% fixed rate mortgage. On the other hand, mortgagors with ARMs get full pleasure of the decreasing interest rates without paying transaction costs.

In the assessment of the individual mortgagor's interest rate risk it is not an issue that the price of the callable FRM gets over 100, as he or she always are allowed to repay at par. Hence, if we assess a mortgagor's interest rate risk out from holding period costs[7], the two loan will be more or less situated equally.

If the interest rate increases there will be major differences. A short rate on 6% (CIBOR 6.33%) corresponding to a 2.3% increase in interest rate level from October 1th, 2001 will reduce prices of BXL-05 to 99.26 and BXL-07 to 98.10. On the other hand the price of 6-32 will fall to 81.54. A mortgagor who is about to sell his house will therefore be much better off with at traditional callable FRM mortgage in case of an increase in interest rates.

The difference is that the traditional FRM affords protection against increasing interest rates 30 years ahead, while the Bolig-X loans protection expires in 4-6 years at a time where the mortgagor has only repayed a small fraction of the principal. For straight ARMs and Bolig-X loans the difference is of course even larger as the mortgagor is completely unprotected to increasing interest rates.

The interesting part about the Bolig-X loan's construction is that the solution is right ahead. Mortgage credit institutions ought to issue Bolig-X loans with a longer time to maturity e.g. 20-30 years. Hence, the mortgagor would receive protection during the total amortization period, and if the house is to be sold after an interest rate increase the bonds could be redeemed at a price way under 100. This is also illustrated in figure 5 where we have included a hypothetical BXL-32, that is a 30 year Bolig-X bond with a cap rate of 7.7%. The price yield curve for this bond is much more similar to the traditional callable FRM. For comparison we have furthermore included model prices for a BXL-6%-32 that is a Bolig-X loan with 30 years to maturity and a cap rate of 6%. The price of this bond is always below the price of the corresponding callable loan, but afford the same protection against interest rate changes. At the same time there will be many situations where the mortgagor pays a lower interest. The callable bonds are normally traded with a spread to the swap curve. This

[7]Holding period costs include the accumulated current payments as well as the market value of the remaining principal. See eg. Jakobsen (1998) for definitions and examples.

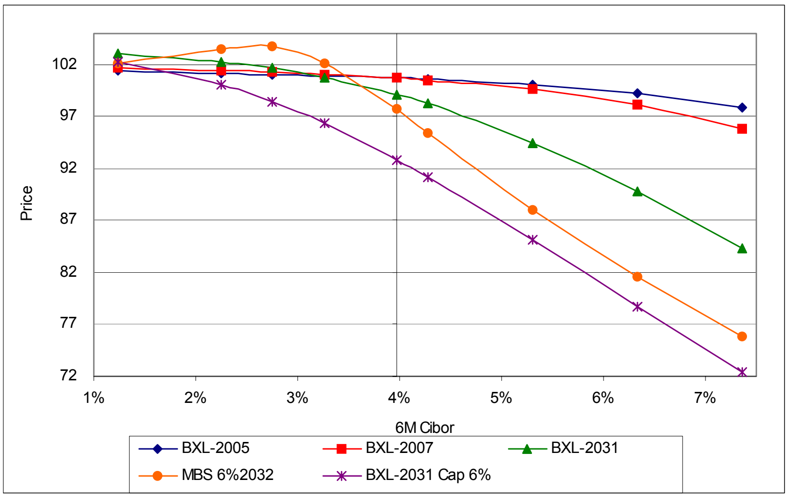

Figure 5: Price yield relationship for Bolig-X loans compared with a traditional 6% Callable Fixed Rate Mortgage.

spread is included in the price of the 6-32[8]. On the other hand we have not included this spread on the Bolig-X loans. If Bolig-X loans are also traded with a spread their prices will decrease and it would be relatively more expensive to finance property using Bolig-X loans[9]. As discussed in Jakobsen & Svenstrup (2000) the primary reason as to why there is a spread is that the investors require a premium to take on risk about mortgagors prepayment behavior. In a Bolig-X loan there is no uncertainty about future prepayment behavior, so from that point of view we could expect a spread much closer to zero.

9 Market valuation of Bolig-X bonds

Market participants are likely to charge a premium due to the liquidity issues and other non-standard issues in Bolig-X bonds. The 10 day average and the fixing prior to the term period are not similar to the standard fixing rules on the Euro market. To estimate the size of the spread that the mortgages trade with we have set up calculations for all trade dates in the period May 10, 2001 to April 2, 2002. The model has been extended to include time dependent parameters and have been calibrated on a daily basis to the swap curve and a set of ATM swaption quotes. In order account for skew in implied volatilities we would have preferred to include out of- and in the money caps or swaptions in the calibration sample. However, these where not available to us.

In Figure 6 we have plotted the market and model prices for the BXL-07.

[8] The spread to the swap curve usually labeled OAS is about 50 bp at the initial interest rate level at 3.7%, which gives a price of 97.71. The spread decreases to 0 at an interest rate level of 7% in aggrement with the method in Jakobsen & Svenstrup (2000). If the spread is set equal to 0 the price of the 6-32 would be 101.95.

[9] Later in the paper we estimate the spread to be somewhere between 20 and 30 basis points.

On the right axis the option adjusted spread has be plotted. The OAS is here defined as the continuously compounded spread that will equate the model and market price when used as an additional discount factor. As we would expect the model prices are higher than the market prices, which gives rise to a positive OAS.

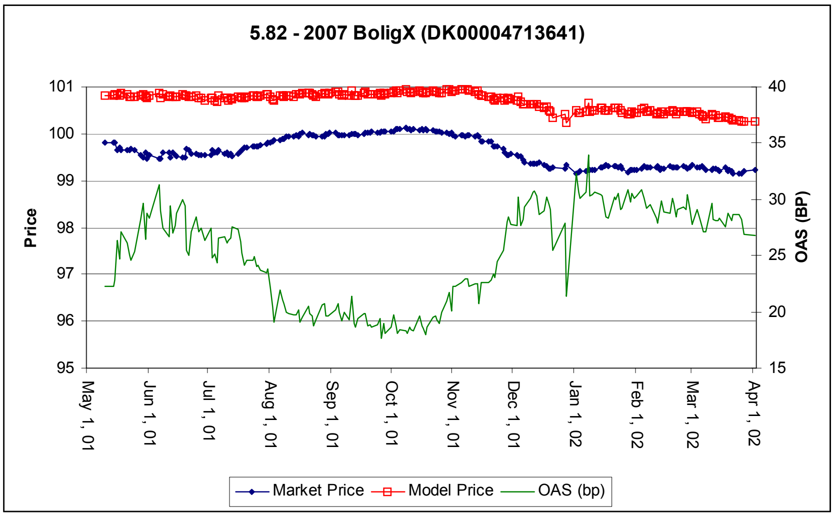

Figure 6: The market and model prices of the 2007 Bolig-X loan with a 7.7% cap rate. The CIR model has been calibrated on a daily basis to the Reuters swap yield curve and a set of swaptions. The option adjusted spread OAS is the additive continuously compounded spread that will equate the model price with the market price.

Table 3 shows summary statistics for the net present value (NPV) which is the difference between the model and market price, the OAS and 6 month CIBOR during the period. The median NPV is 1.11 price point and the 25% and 75% percentiles are 0.89 and 1.2 price point. OAS is another way of expressing the NPV, as it is the yield required to amortize the NPV over the remaining life of the bond. Hence, these two measures are highly correlated with a coefficient of 0.97. The OAS has been between 18 and 34 bps during the period and with an average of 25. This seems to be a reasonable spread compared to similar products on the Euro market but maybe on the large side.

	Total Obs	Mean	Median	Max	Min	Percentile 25%	Percentile 75%
NPV	221	1.07	1.11	1.41	0.77	0.89	1.2
OAS (bp)	221	25	26	34	18	21	29
CIBOR 6M %	221	4.13	3.82	5.17	3.48	3.65	4.73

Table 3: Summary statistics for the net present value NPV, the option adjusted spread (OAS) and the 6 month CIBOR during the period May 10, 2001 to Apr 2, 2002.

An ARM settled continuously at the short rate would have a zero duration. However, in case of a lower fixing frequency any fixed payments will have a duration corresponding to the time to their maturity, a result well-known from zero coupon bonds. Therefore, the duration of the Bolig-X loans will depend on the number of fixed coupons, which could be from 1 to 3 and the time to the next term date. For the Bolig-X bonds with caps the duration is also effected by the size of the yield premium and the embedded caps. Figure 7 shows the Krone Duration and the Krone Convexity of the BXL 07 bond[10] as well as the 6 Month CIBOR rate. Notice that the embedded cap is out-of-the money as the CIBOR rate have been less than 5.17% during the period.

Maybe the most interesting observation that can be made from figure 7 is the development of the duration as time passes. In the middle of May 2001 three payments have been fixed with a maturity of roughly 1, 4 and 7 months(this portfolio would approximately have a duration of (1+4+7)/12) 1 year. As time passes the duration of these payments decrease until the beginning of the next fixing period Nov 1, 2001, where only one payment (with maturity 2/12=0.17) is left. Each day of the fixing period a tenth of the next coupon rate is fixed. This is easily seen in the figure, as the duration increases from the beginning to the end of the fixing period. Ignoring the 10 day average during the fixing period would therefore cause durations to be seriously distorted. A study of the importance of index dynamics for the interest rate sensitivity of ARMs can be found in e.g. Stanton & Wallace (1999).

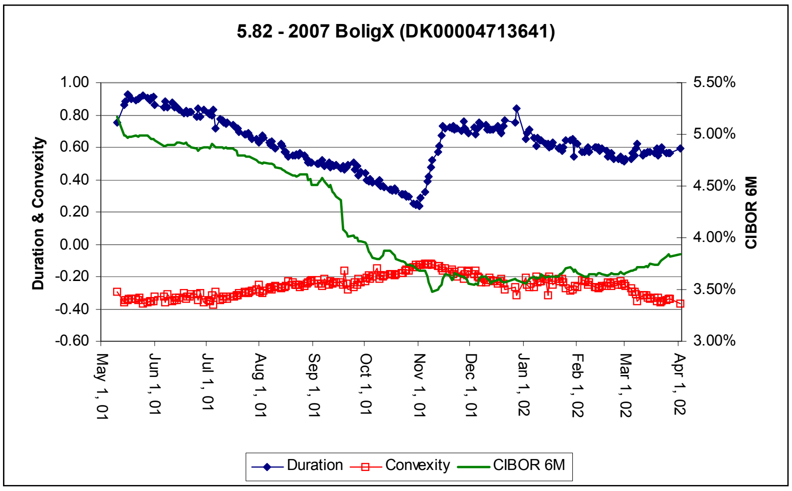

Figure 7: This figure shows the Krone Duration (Dollar Duration) and the Krone Convexity (Dollar Convexity) on the left axis. On the right axis the development in the 6 Month CIBOR is shown.

Tests show that ignoring the 10 day average when fixing future payments in

[10]The Krone Duration and Convexity are here defined as the Krone sensitivity to a discretely compunded parallel shift to the entire yield curve. Both figures are computed from the usual finite difference approximation. Krone or Dollar durations refer to the fact that it is an absolute sensitivity and not a relative sensitivity. To get the relative duration divide by the dirty-price. We use market practice of calculating the duration keeping the OAS fixed.

the grid has almost no effect on price (see also Appendix).

10 Design of mortgages

In this section we discuss the design of Bolig-X loans and look at aspects related to counselling of house holders in their mortgage choice. We disregard costs and contribution fees and focus primarily on the payments before tax.

Our main suggestion is to issue 30 year Bolig-X bonds and hence get the interest cap during the entire amortization period. The existing Bolig-X bonds and hence the caps mature in 4 to 6 years. In 4 to 6 years the mortgage holder will only have repaid a few percent of the initial principal and as illustrated in figure 8 the value of the remaining principal will almost be par for all interest rate levels. Even though the Bolig-X loans right now include a better protection than the straight ARMs, the protection disappears rapidly in a few years.

Worst case scenario is of course an increase in interest rates combined with falling prices of housing. Households facing a borrowing constraint will not be able to pay the mortgage but on the other hand they will not be able to sell their house without loss. It is also too late to refinance into FRMs as the payments in these bonds would be at an even higher level after an increase in interest rates. In this situation compulsory sales would flourish with a further pressure on house prices[11].

On the other hand if the mortgagor had issued an ARM with an interest rate cap, then as long as the household can meet the payment corresponding to the cap rate he or she could stay. Even if the household chooses to sell, the cap, cf. figure 8, will mean that the price of loan will be below 100, so it is likely that the household can repay the debt even with decreasing house prices.

Everything equal mortgagors will prefer a long cap on a low level. On the other hand investors will charge for this cap by lower prices and hence higher payments. As it will be evident from below the mortgagors have to weigh low payments right now against high safety.

Figure 9 shows model prices for Bolig-X bonds with a maturity of 5, 10, 15, 20 and 30 years and cap rates on 6, 7, and 8%. The underlying mortgages are all 30 year. In order to compare we also included the price of 30 year Bolig-X loan without cap and a 6% callable FRM computed with and without a option adjusted spread. In all calculations the first coupon has been set equal to the level on October 1, 2001, that is 3.97%. The three Bolig-X loans with a cap all include a 20 bp yield premium.

As expected a lower cap rate will result in lower prices e.g. 6% to 8%. The same holds for a longer maturity of the bond and hence a longer cap. Anyway, the difference between a 30 year Bolig-X loan without cap and a Bolig-X with an 8% cap is less than 1 point. If one wants a 6% cap rate it will cost approximately 7.5 price points. In both cases the mortgagor pays an additional 20 bp yield premium. It is interesting that there is less than 1 point difference between getting a 20 and 30 year interest rate cap. This is among other things due to amortization of the principal which is very fast in the last period of the loan.

[11] An alarming feature is that in these days it is argued that house prices are increasing because of the ARMs. We fear that the longer time we have low rates the more reckless the households and their counsellors get and the smaller increase in interest rates is sufficient to ruin the housing market.

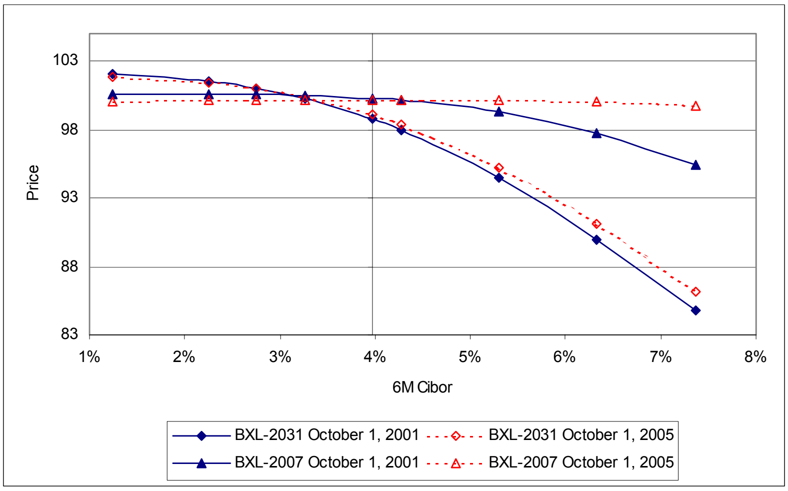

Figure 8: Price movements over time for two Bolig-X bonds. The two full lines show the price yield on October 1, 2001 while the dotted lines show the prices 5 year ahead. This figure illustrates how rapidly the interest rate protection decreases for the BLX-07 bond.

If we compare it to the 30 year 6% callable FRM it will cost 97.72 if we add a spread corresponding to the market spread October 1, 2001, while the price without spread is 101.95. That is, even with the spread the callable FRM is more expensive than a Bolig-X loan with a 6% cap, which reflects the fact that investors receive the full 6% in coupon together with the irrational prepayment behavior in case of decreasing interest rate.

In table 4 and figure 10 we have calculated the actual and maximal payments for the given loans. All loans have a proceed of 1 million DKK. We have disregarded all costs and contributions. For the Bolig-X loans we take a CIBOR rate of 3.97% as of October 1th, 2001 as starting point[12]. If the mortgagor chooses a loan with a 6% cap rate she will have to make a monthly payment of 5,256 DKK. If interest rates increase the maximal payment is 6,471 DKK. In case a 8% cap rate is chosen the bonds can be sold at a higher price and the payment right now will be 4,889. In return, the maximal payment before tax is now 7,374 DKK per month. The cheapest mortgage right now is of course the Bolig-X without cap, which corresponds roughly to a standard Danish ARM (F1). Here there is no ceiling over future payments. Finally, there is the traditional callable FRM.

[12]The mortgage coupon on January 1, 2002 has been set at 5.36% but in the calculations we have chosen to compute the current payment from the interest rate level as of October 1, 2001 as is given by a short rate of 3.7% corresponding to a 6 month CIBOR of 3.974%. We have used 121 quarterly term dates. With a quarterly coupon rate of $(3.974+0.2)/4 =$ 1.04351% and 121 terms the quarterly payment per 1 mil. DKK in principal is 14,590. For a 30 year Bolig-X loan with a 6% cap the price is 92.535. To receive a proceed of 1 mil. DKK the first quarterly payment would be $14{,}590 \cdot 100/92.535 = 15{,}767$, corresponding to 6256 DKK a month. The maximal payment is computed in the same way just with a quarterly coupon of 6%/4=1.5% in stead of 1.04351%. Notice, that in periods with a coupon rate below the maximum rate the debt is repayed faster, which subsequent would reduce the maximal payment.

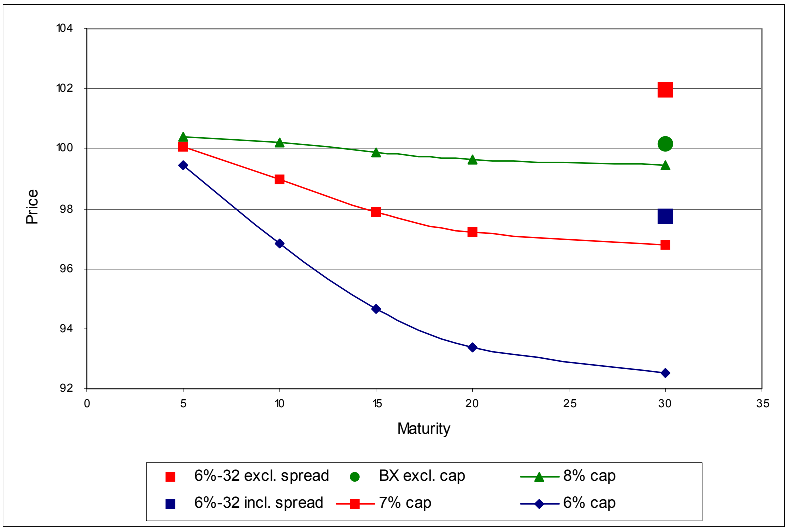

Figure 9: Model prices for bonds depending on cap rate and maturity.

This mortgage has the highest mortgage right now but in return this payment cannot increase, and it is furthermore the mortgage with the lowest maximal payment.

At the moment all historical experiences are disregarded and all mortgagors are advised to issue ARMs. Just a few years ago the advice was that a household should at least afford to finance the house using traditional callable FRMs. If this was the case then maybe an ARM could be considered. Even this piece of advice is problematic as the ARM could show to be more expensive than the alternative FRM that originally was rejected.

If the market offers a range of loans with lifetime cap rates the counselling of mortgage holders would be much concrete. A mortgagor could use the maximal possible payment as reference point. Given this maximal payment there will be a tradeoff between proceeds and the initial payment, such that a higher proceed will require a less risky mortgage. This is completely in line with the US litterature discussed in the introduction.

If we focus on the payments before taxes there is a relatively small difference between mortgages with a 6% and 8% cap rate. The difference is larger in an after tax consideration as the loan with a 6% cap is issued as a bond loan at price 92.5, that is with a loss due to depreciation that is not tax-deductable. In practice one would either issue the bonds as cash loans or issue bonds with a higher cap rate e.g. 7% or 8%[13].

[13]It is also possible to increase the yield premium to e.g. 60 bp. This would increase the price from 92.53 to 94.27. The problem is that an increased yield premium does not have an effect in those scenarios where the cap already is binding, so that a very large yield premium is required in order increase the price.

On the other hand as the yield premium works as a fixed payment during the entire maturity of the bond it increases the mortgagors minimum payment and hence the price risk when interest rates decrease.

The maximal price is obtained at a premium of 6%, that is the mortgagor will never pay less

Loan Type	Maturity in Years 5	10	15	20	30
6% cap	4,891	5,023	5,137	5,209	5,256
7% cap	4,859	4,914	4,967	5,002	5,024
8% cap	4,845	4,854	4,871	4,882	4,889
6% cap Max	6,022	6,185	6,326	6,414	6,471
7% cap Max	6,642	6,717	6,790	6,837	6,868
8% cap Max	7,307	7,321	7,346	7,363	7,374
6-32 incl. spread	-	-	-	-	6,128
6-32 excl. spread	-	-	-	-	5,874
BX no cap	-	-	-	-	4,740

Table 4: Monthly payments before taxes for a mortgage with a proceed of 1 mio. DKK depending on cap rate and maturity of cap. The current payments are only known until next refinancing date. This means that only the 30-year bonds have a life time cap.

11 Discussion

In our opinion long ARMs with life time caps have the potential to create a seminal innovation of the Danish mortgage market. In contrast to the existing straight ARMs these bonds put a necessary ceiling on the mortgage payments. Allowing ordinary households to buy houses on the borderline of their financial capabilities and only financing the first year of a 30 year amortization period is in our opinion totally irresponsible. With a life time cap the maximal payment is guaranteed and the calculations show that the payments will only be marginally higher than the unsecured loan. The risk has not disappeared but it has been transferred against charge from the mortgagor to the financial investors in the money- and bond markets. Furthermore, the existence of the cap increase the value of the delivery option.

Compared to traditional Danish ARMs with interest rate guaranties the Bolig-X product is simpler, more easily clarified in relation to taxes[14] and probably also cheaper. At least they sell better.

In relation to the traditional callable FRMs the Bolig-X bonds give similar insurance when interest rates increase and automatic payment reduction when they fall. Mortgagors avoid transaction costs when prepaying and investors are able to cover their interest rate risk using standard interest rate products without worrying about modelling prepayment behavior and other difficulties related to the traditional FRMs.

Just as for callable FRMs there is a risk of illiquid bond series. If the level of interest rates gets close to or higher than the cap rate the prices will drop too far below 100 and hence prevent new issues in that series. In case of a lower interest rate, mortgagors are likely to prefer series closer to the money. However, old series could be reopened if interest rates return to the initial level. In contrast to existing callable FRM series these are not required to be closed

than 6%. But that is exactly a 30 year non callable loan - which is something no household wants to issue.

[14] There has been issues with tax authorities but according to Totalkredit these issues have been settled.

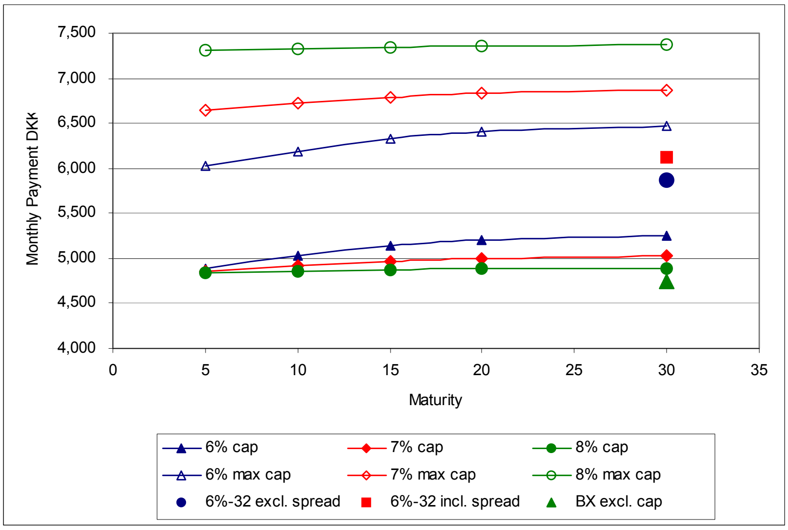

Figure 10: Actual and maximal payments before tax as a function of maturity for loans with a proceed equal to 1 million DKK. Series with full marks represent the current payment and the dotted marks the corresponding maximal possible payments.

for new issues when prices exceed 100.

Illiquid bonds at prices below 100 could mean that mortgagors have to buy at a premium in order to redeem their loans. This is also the situation for traditional callable FRMs. With an adjustable rate it is unlikely that mortgagors risk redeeming their bonds at prices above 100. Under all circumstances this type of bonds will not depend upon the behavior of the borrowers and the individual mortgage institutions series will be close substitutes.

In the sketched bond type the interest rate cap cannot be changed. In practice it could happen by a simple renewal of the loans. After an increase in interest rates the mortgagor could redeem his old loan at market price and issue a new bond with a higher cap rate. After an interest rate decrease the mortgagor could against an additional payment take out a new loan with lower cap rate.

As an alternative to these prepayments the institutions could choose to embed an automatically adjustment in the loan. In the US market ARMs are issued with so called rolling caps as well as lifetime caps.

A traditional problem with adjustable rate mortgages is to find a stable index rate. It won't be a problem that Totalkredit's bonds with a nominal of 25 billion DKK is linked to 6 month CIBOR, but if the mortgage credit institutions are to fix 3-400 billion DKK on CIBOR in too short periods every year, then a tremendous focus would be put on the banks that report CIBOR. Similarly the liquidity in traditional F1 ARMs would probably drop dramatically if RD offered loans indexed after the F1-rate but also had 10, 20 or 30 year caps.

The starting point for our calculations is that long Bolig-X like bonds are priced in line with or better than traditional callable FRMs. It is naturally an open question whether the market is willing to absorb large issues of Bolig-X like series even though a corresponding amount of the traditional bonds are repaid.

All these considerations are of course of hypothetical nature and the proposal to Totalkredit and other mortgage credit institutions is simple to try. The product could die quietly but it could also turn the Danish mortgage market upside down. Apparently there are lots of mortgagors who dare to refinance their house once a year in order to obtain a lower payment right now, but it is our assessment that even more are willing to pay a bit extra if they can budget with a maximal mortgage payment the next 30 year.

12 Conclusion

In the Danish mortgage market mostly straight ARMs have been issued. However, the US mortgage choice literature indicates that hybrid ARMs are attractive to mortgagors and the experiences from the Bolig-X bonds support this. Furthermore, our calculations indicate that the limited effect on monthly payments of buying life-time out-the-money caps, will be of interest for many of the mortgagors currently rolling over their 30 year mortgages at the 1 year rate. It appears very reasonable that risk averse mortgagors will be willing to insure themselves against worst case losses. The more likely it is that interest rates are going to increase, the more expensive it will be to get a life time ceiling over the mortgage payment. However, this is not a reason to issue a straight ARM - on the contrary.

Also of interest to e.g. the US mortgage market, we argue that the delivery option is a very important and efficient means to ensure a tighter match between the assets and liabilities in a household portfolio. Furthermore, a by-product of the delivery option is an increase in the mobility of the labour force as a whole, which could also be of macro-economic importance.

References

Aadu, J. & Sirmans, C. (1995), 'Differentiated contracts, heterogeneous borrowers, and the mortgage choice decision', *Journal of Money, Credit and Banking* **27**(2), 498–510.

Bondorf, S., Sørensen, D. & Carlsen, M. (2000), 'Rentesikring - nyt instrument til afdækning af likviditetsrisiko på rentetilpasningslån', *FINANS/INVEST* **3**.

Brueckner, J. K. & Follain, J. R. (1988), 'The rise and fall of the arm: An econometric analysis of mortgage choice', *The Review of Economics and Statistics* **70**(1), 93–102.

Campbell, J. Y. & Cocco, J. F. (2002), 'Household risk management and optimal mortgage choice', *Working Paper, Harvard University and London Business School* .

Cox, J. C., Ingersoll, J. E. & Ross, S. A. (1985), 'A theory of the term structure of interest rates', *Econometrica* **53**(2), 385–408.

Dhillon, U. S., Schilling, J. D. & Sirmans, C. (1987), 'Choosing between fixed and adjustable rate mortgages:note', *Journal of Money, Credit and Banking* **19**(2), 260–267.

Hull, J. & White, A. (1993), 'Efficient procedures for valuing european and american path-dependent options', *The Journal of Derivatives* **1**(1), 21–31.

Jakobsen, S. (1992), Prepayment and the Valuation of Mortgage-Backed Bonds, PhD thesis, The Aarhus School of Business.

Jakobsen, S. (1995), 'A mixture distribution approach to the valuation of mortgage backed securities', *Working Paper, The Aarhus School of Business* .

Jakobsen, S. (1998), 'Rådgivning om realkreditlån', *FINANS/INVEST* **3**.

Jakobsen, S. & Rasmussen, N. (1999), 'Prisfastsættelse af konverterbare obligationer', *FINANS/INVEST* **8**.

Jakobsen, S. & Svenstrup, M. (1999), 'Modeller for konverteringsadfærd', *FINANS/INVEST* **7**.

Jakobsen, S. & Svenstrup, M. (2000), 'Sætter investorerne pris på konverteringsrate risiko', *FINANS/INVEST* **2**.

Jakobsen, S. & Svenstrup, M. (2001), 'Variabel rente med loft - det perfekte realkreditlån', *FINANS/INVEST* **7**, 18–25.

Leroy, S. F. (1996), 'Mortgage valuation under optimal prepayment', *Review of Financial Studies* **9**(3), 817–844.

Mortgage Financing in Denmark (1999), www.realkreditraadet.dk.

Nielsen, S. & Poulsen, R. (2002*a*), 'Planning your own debt', *European Financial Management* **8**(2), 193–210.

Nielsen, S. & Poulsen, R. (2002*b*), 'A two-factor stochastic programming model of danish mortgage-backed securities', *forthcomming - Journal of Economic Dynamics and Control* .

Pedersen, B. W. (2000), 'Ny realkredit model - med fokus på gennemsigtighed', *FINANS/INVEST* **6**.

Richard, S. & Roll, R. (1989), 'Prepayments on fixed-rate mortgage backed securities', *Journal of Portfolio Management* pp. 73–82.

Schwartz, E. & Torous, W. (1989), 'Prepayment and the valuation of mortgage-backed securities', *The Journal of Finance* **XLIV**(2), 375–392.

Stanton, R. & Wallace, N. (1998), 'Mortgage choice: What's the point?', *Real Estate Economics* **26**, 173–205.

Stanton, R. & Wallace, N. (1999), 'Anatomy of and ARM: The interest-rate risk of adjustable-rate mortgages', *Journal of Real Estate Finance and Economics* **19**(1), 49–67.

Svenstrup, M. (2002), 'Valuation of path-dependent interest rate derivatives in a finite difference setup', *Working Paper, The Aarhus School of Business* .

Thomsen, L. & Tørnes-Hansen, S. (2000), 'Rentegarantier - nye perspektiver for lån med rentetilpasning', *FINANS/INVEST* **7**.

Tørnes-Hansen, B. G. L. . S. (1997), 'FlexLån TM - fremtidens realkredit', *Finans/Invest* **2**, 9–15.

Wilmott, P., Dewynne, J. & Howison, S. (1993), *Option Pricing: Mathematical Models and Computation*, Oxford Financial Press.

A Appendix

Tests show that ignoring the 10 day average when fixing future payments in the grid has almost no effect. The length of the fixing period is simply to short relative to the time between two fixings in order to have a significant impact of the index volatility. To illustrate this consider for example the variance of an average in a Vasicek model

$$dr_t = \kappa (\mu - r_t) \, dt + \sigma dW_t.$$

If we let $A_T = \frac{1}{n} \sum_{i=1}^{n} r_{t_i}$ it is easily seen using the property of independent increments that

$$\begin{aligned}
Var(A_T) &= Var\left(\frac{1}{n}\sum_{i=1}^{n} r_{t_i}\right) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} r_{t_i}\right) \\
&= \frac{1}{n^2} Var\left(n r_{t_1} + (n-1)(r_{t_2} - r_{t_1}) + .. + 1\left(r_{t_n} - r_{t_{n-1}}\right)\right) \\
&= \frac{1}{n^2} Var\left(\sum_{i=1}^{n} (n-i+1)\left(r_{t_i} - r_{t_{i-1}}\right)\right) \\
&= \sum_{i=1}^{n} \frac{(n-i+1)^2}{n^2} Var\left(r_{t_i} - r_{t_{i-1}}\right) \\
&= \sigma^2 \sum_{i=1}^{n} \frac{(n-i+1)^2}{n^2} (t_i - t_{i-1}) \\
&= \sigma^2 \sum_{i=1}^{n} \frac{(n-i+1)^2}{n^2} \Delta_i.
\end{aligned}$$

Hence if the Δ_1 is large relative to the rest of the intervals the decrease in variance is small. In the case of the Bolig-X loans even for the first fixing there will be 6 months and the fixing period is 10 days then $\Delta_1 = 0.5$ and $\Delta_i = 1/252$ hence the period from t_0 to t_1 contributes with 98% of the total variance of A_T. For following fixing periods this is even more pronounced as Δ_1 is even larger while other Δ_i's remain the same. Of course this argument does not hold as we approach a fixing period, but then the value of the cap is not as sensitive to the volatility.

Efficient Control Variates and Strategies for Bermudan Swaptions in a Libor Market Model

by

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Abstract

This paper concerns the problem of valuing Bermudan swaptions in a Libor market model. In particular we consider various efficiency improvement techniques for a Monte Carlo based valuation method. We suggest a simplification of the Andersen (2000) exercise strategy and find it to be much more efficient. Furthermore, we test a range of control variates for Bermudan swaptions using a control variate technique for American options proposed in Rasmussen (2002). Application of these efficiency improvements in the Primal-Dual simulation algorithm of Andersen & Broadie (2001) improves both upper and lower bounds for the price estimates. For the Primal-Dual simulation algorithm we examine the variance-bias trade-off between the numbers of outer an inner paths. Finally, we demonstrate that the presence of stochastic volatility increases the expected losses from using the simple strategy in Andersen (2000).

JEL Codes: G12; G13; E43;

Keywords: Bermudan Swaptions; Control Variates; Exercise Strategy; Primal-Dual Algorithm; Stochastic Volatility;

1 Introduction

For a long time valuation of options with early exercise features or other free boundary problems by simulation was considered impossible. Researchers and practitioners have been focusing on lattice methods such as trees and finite difference methods whenever these American style contingent claims were encountered. It is well known that lattice methods suffer from a curse of dimensionality.

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This means that valuation of assets with payoffs depending on more than three state variables is considered to be unfeasible. This is the case for options on several assets, interest rate models with a large number of state variables, and models driven by several factors.

A key contribution to the solution to this problem was given in Broadie & Glasserman (1997), where a simulation algorithm providing asymptotically unbiased upper and lower bounds on the option value was presented. The problem with this method is that it is difficult to generalize and requires that there are few exercise decisions. However, recently new algorithms have been developed.

Haugh & Kogan (2001) present the value of an American option as the solution to a dual minimization problem over all super martingales, so that any given super martingale will result in an upper bound for the option. Andersen & Broadie (2001) recognize that the use of a martingale instead of a submartingale in the dual problem expressed in Haugh & Kogan (2001) will result in tighter upper bounds and they propose an improved simulation algorithm. A primal solution algorithm that provides an exercise strategy and hence an estimate of the lower bound is input to this new Primal-Dual simulation algorithm. Any primal algorithm can be applied, for example Least-Square Monte-Carlo (Longstaff & Schwartz (2001)), nonparametric methods (Andersen (2000)), low-dimensional lattice methods (Longstaff, Santa-Clara & Schwartz (2001*b*)), and others.

Andersen & Broadie (2001) demonstrate the simplicity and efficiency of the simulation algorithm in several examples, including multi-asset equity options and Bermudan swaptions. However, only little emphasis was placed on numerical efficiency and variance reduction techniques.

For the purpose of pricing American style contingent claims in a Monte Carlo framework Rasmussen (2002) develops an extension of traditional control variate techniques, in which sampling of the controls occurs at the time of exercise of the American option and he demonstrates the efficiency in the multi-asset equity option case.

The primary contributions of our paper are the following. First, we demonstrate that a simplification of the exercise strategies proposed in Andersen (2000) proves much more efficient in the Bermudan swaption case. Due to the structure of Bermudan swaptions, the most valuable core European swaption will almost always be the first to mature, and we show that ignoring the rest is computationally more efficient but results in the same prices. Secondly, we test the efficiency of a range of control variates with respect to Bermudan swaptions using the Rasmussen (2002) sampling algorithm. We illustrate how to apply it using dividend paying assets. Thirdly, we consider the rate of convergence of the Primal-Dual algorithm with an emphasis on the variance-bias trade-off due to nested simulations. When combining the simplified strategies and the control variates with a better variance-bias trade-off we experience significant improvements, and more specifically we get lower duality gaps and lower standard deviations in less time.

Finally, we implement a version of the Libor market model with stochastic volatility in order to examine the effect on Bermudan swaptions. As expected.

we find that the duality gaps are increasing in variance of the volatility and decreasing in mean reversion. Our work is also related to Bjerregaard Pedersen (1999) who applies the Broadie & Glasserman (1997) method to Bermudan Swaptions.

The general model setup, notation and problem statement are laid out in the first section. Furthermore, we briefly summarize theoretical results required in the rest of the paper. In section 3 and section 4 we briefly introduce the Primal-Dual simulation algorithm and some variance reduction techniques. The Libor market model and the assets, that are considered in this paper, are introduced in section 5. Numerical results are found in section 6, and finally, we make our conclusions in section 7.

2 The Optimal Stopping Problem

We assume that we have a dynamically complete financial market in which all uncertainty is described by a standard filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P}\right)$. The information set $\mathcal{F}_t$ is defined as the natural filtration generated by a multi-dimensional Wiener process which drives all asset prices until some fixed time T augmented with the usual null sets. We assume the existence of a $\mathbb{P}$ equivalent probability measure $\mathbb{Q}$ under which asset prices discounted with the numeraire asset $B(t)$ are martingales. Let $E_t^{\mathbb{Q}}(\cdot) = E_t^{\mathbb{Q}}(\cdot|\mathcal{F}_t)$ denote the expectation conditional on information available at time t. Hence, the time t value of any asset $Q(t)$ is given by

$$\frac{Q(t)}{B(t)} = E_t^{\mathbb{Q}}\left(\frac{Q(s)}{B(s)}\right),\ t \leq s.$$

American options are options with an early exercise feature and are characterized by the payoff $h(t)$ paid upon exercise and the set of dates $\mathcal{T}$ on which the holder is allowed to exercise. American options, which can only be exercised on a discrete set of dates, are often referred to as Bermudan options.

The problem of valuing an American option is usually posed as an optimal stopping problem. Following Andersen & Broadie (2001) and Haugh & Kogan (2001), we will denote the following the *primal problem*

$$\frac{Q(t)}{B(t)} = \sup_{\tau\in\Gamma(t)} E_t^{\mathbb{Q}}\left(\frac{h(\tau)}{B(\tau)}\right) \tag{1}$$

where $\Gamma(t)$ denotes the set of optional stopping times τ taking values in $\mathcal{T}(t) = \mathcal{T} \cap [t, T]$, T being the maturity of the option. Notice that any optional[1] stopping time belonging to $\Gamma(t)$ will generate a lower bound on the true option price.

[1] We refer to stochastic calculus textbooks (e.g. Protter (1995)) for a definition of optional stopping times, and we assume that all stoppings considered are optional for the given processes in question.

Andersen & Broadie (2001) define the *dual problem* to (1) by first noting that a valid upper bound can be found by using any adapted martingale π_t

$$\begin{aligned}\frac{Q(0)}{B(0)} &= \sup_{\tau\in\Gamma(0)} E_0^{\mathbb{Q}}\left(\frac{h(\tau)}{B(\tau)}+\pi_\tau-\pi_\tau\right) &(2)\\ &= \pi_0+\sup_{\tau\in\Gamma(0)} E_0^{\mathbb{Q}}\left(\frac{h(\tau)}{B(\tau)}-\pi_\tau\right)\\ &\le \pi_0+E_0^{\mathbb{Q}}\left(\max_{t\in\mathcal{T}(0)}\left(\frac{h(t)}{B(t)}-\pi_t\right)\right). &(3)\end{aligned}$$

The second equality follows from the *Optional Sampling Theorem* and the martingale property of π. As (3) holds for any martingale, it follows that

$$\frac{Q(0)}{B(0)} \le \inf_{\pi}\left(\pi_0+E_0^{\mathbb{Q}}\left[\max_{t\in\mathcal{T}(0)}\left(\frac{h(t)}{B(t)}-\pi_t\right)\right]\right). \tag{4}$$

This definition of the upper bound is slightly different to the one proposed in Haugh & Kogan (2001). They define the *dual problem* over all super martingales which, of course, includes all martingales. This generates more conservative upper bounds, as also noted in Footnote 1 of Andersen & Broadie (2001). This can easily be seen as the optional sampling theorem for super martingales adds an extra inequality in the derivation of the upper bound in equation (2) as $\pi_0 \ge E_0^{\mathbb{Q}}(\pi_\tau)$ when π is a supermartingale.

It can be proved that, when choosing π in equation (2) as the martingale part of the Doob-Meyer decomposition of the deflated price process $Q(t)/B(t)$, equation (2) holds with equality. As a result, Andersen & Broadie (2001) propose to use the martingale part of a discounted lower bound price $L(t)$ as a proxy for the true value process. The lower bound price is defined by

$$\frac{L(t)}{B(t)} = E_t^{\mathbb{Q}}\left(\frac{h(\tau_t)}{B(\tau_t)}\right),$$

where τ_t is an optional stopping time given by some exercise strategy used from t and onward. Having defined the lower bound process we use the following π martingale process

$$\pi(t_1) = \frac{L(t_1)}{B(t_1)}$$

and for exercise dates $t_2, ..., t_d$

$$\pi_k = \pi_{k-1}+\frac{L(t_k)}{B(t_k)}-\frac{L(t_{k-1})}{B(t_{k-1})}-I(t_{k-1})\,E_{t_{k-1}}^{\mathbb{Q}}\left[\frac{L(t_k)}{B(t_k)}-\frac{L(t_{k-1})}{B(t_{k-1})}\right]. \tag{5}$$

With this choice of π process, an upper bound is seen to be the lower bound plus the value of a non-standard lookback option D_0, which is denoted the *duality gap*

$$\frac{Q(0)}{B(0)} \le \frac{L(0)}{B(0)}+E_0^{\mathbb{Q}}\left(\max_{t\in\mathcal{T}(0)}\left(\frac{h(t)}{B(t)}-\pi_t\right)\right) = \frac{L(0)}{B(0)}+D_0.$$

3 The Primal-Dual Simulation Algorithm

We apply the simulation algorithm laid out in Andersen & Broadie (2001) and refer to their paper for a thorough discussion of the entire algorithm. However, a short outline is appropriate. The algorithm generates estimates of the duality gap D_0 by simulating the π process and the discounted payoff process $h(t)/B(t)$. The main difficulty is determining the $L(t)/B(t)$ terms in the π process given by equation (5.). This is effectively done by running nested simulations replacing $L(t_k)/B(t_k)$ and $E_{t_k}^{\mathbb{Q}}\left(\frac{L(t_{k+1})}{B(t_{k+1})}\right)$ with Monte Carlo estimates based on m nested simulations $\frac{L(t_k)}{B(t_k)} + \varepsilon_k$ and $E_{t_k}^{\mathbb{Q}}\left(\frac{L(t_{k+1})}{B(t_{k+1})}\right) + \varepsilon_k^{'}$ respectively. $\varepsilon_k, \varepsilon_k^{'}$ are the mean zero simulation errors. By summing noise terms in π it follows that the Monte Carlo estimate $\hat{\pi}$ of the exact π process is

$$\hat{\pi}_k = \pi_k + \tilde{\varepsilon}_k,$$

where the noise term $\tilde{\varepsilon}_k$ is a sum of the mean zero noise terms. It can easily be proved that the noisy estimates of the π process make the estimate of the duality gap D_0 upward biased. Hence, the Monte Carlo estimator of the duality gap $\hat{D}_0$ using n "outer" simulations and m inner simulations, is

$$\hat{D}_0(n,m) = \frac{1}{n}\sum_{i=1}^{n} \max_{1\leq k\leq d}\left(\frac{h^i(t_k)}{B^i(t_k)} - \hat{\pi}_k^i\right).$$

Using both the upper and lower bound we can construct a somewhat conservative confidence interval for the price estimate based on an $n \times m$ simulation trial, as

$$\left[\hat{L}_0(n) - z_{1-\frac{\alpha}{2}}\frac{\hat{s}_L(n)}{\sqrt{n}}, \hat{L}_0(n) + \hat{D}_0(n,m) + z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{s}_L^2(n)}{n} + \frac{\hat{s}_D^2(n,m)}{m}}\right] \quad (6)$$

where z_x denotes the xth percentile of the standard normal distribution and $\hat{s}_L$ and $\hat{s}_D$ are the sample standard deviations.

4 Variance Reduction

Andersen & Broadie (2001) do not consider variance reduction techniques. In this paper we test the effect of antithetic variates (AS) and control variates (CV) see e.g. Glynn (1994). A more elaborate discussion of variance reduction techniques applied in a finance setting can be found in e.g. Boyle, Brodie & Glasserman (1997).

4.1 Antithetic Variates

The method of antithetic variates is widely used and is based on the simple observation that if ε has a standard normal distribution so does $-\varepsilon$. The idea is that random inputs obtained from the collection of antithetic pairs $\{(\varepsilon_i, -\varepsilon_i)\}_{i=1}^{I}$ are more regularly distributed than a collection of $2I$ independent samples. However, antithetic variates only work when the discounted payoffs estimated from ε_i and $-\varepsilon_i$ are negatively correlated - increasing efficiency with correlation. As a result, antithetic variates work very well for linear integrands and fails in the case of symmetric integrands. For simple options, determining the efficiency of antithetics beforehand can sometimes be done. However, when dealing with path-dependent American options, this is not possible.

4.2 Control Variates

The method of control variates is based on the principle 'use what you know'. The most straightforward implementation of control variates replaces the evaluation of an unknown expectation with the evaluation of the difference between the unknown quantity and another expectation whose value is known. Suppose we at time t know the expectation $E_t^{\mathbb{Q}}[Y]$ of an $M-$dimensional stochastic variable Y. Assuming that we can sample I realizations of a scalar variable Z exactly, an unbiased estimator $\bar{Z}^{CV}$ of $E_t^{\mathbb{Q}}[Z]$ is given by

$$\bar{Z}_t^{CV}(\beta) = \frac{1}{I}\sum_{i=1}^{I}\left(Z^i - \beta'\left(Y^i - E_t^{\mathbb{Q}}[Y]\right)\right) \tag{7}$$

with variance

$$Var\left(\bar{Z}_t^{CV}(\beta)\right) = \frac{1}{I}\left(\sigma_Z^2 - 2\beta'\Sigma_{YZ} + \beta'\Sigma_Y\beta\right) \tag{8}$$

for some appropriately chosen vector $\beta \in \mathbb{R}^M$. Here σ_Z^2 is the variance of Z, Σ_Y denotes the covariance matrix of the controls and Σ_{YZ} is the vector of covariances between Z and Y.

The variance minimizing choice of β is given by

$$\beta^* = \Sigma_Y^{-1}\Sigma_{YZ}. \tag{9}$$

Inserting (9) into (8) we have at optimality

$$Var\left(\bar{Z}_t^{CV}(\beta^*)\right) = \frac{1}{I}\left(1 - R^2\right)\sigma_Z^2,$$

where

$$R^2 = \frac{\Sigma'_{YZ}\Sigma_Y^{-1}\Sigma_{YZ}}{\sigma_Z^2}$$

Thus, effectiveness is determined by the size of the coefficient of multiple correlation R between Z and the control variates Y. In addition we notice that since $R^2 \in (0, 1)$ using the variance-minimizing coefficient β^*, we are guaranteed not to increase variance.

4.2.1 Control Variates for American options

It is not immediately clear how control variates may be applied to American style options. A naive guess would be to sample the controls at fixed times in the exercise period. In a recent paper, Rasmussen (2002) illustrates that the control sampled at the time of exercise of the Bermudan option, has much higher correlation with the discounted payoff from the option than with the control sampled at e.g. expiry of the option. It is shown in particular, that for any given martingale process Y and optional stopping times τ, σ for which $t \leq \tau \leq \sigma \leq T$ $\mathbb{Q}$-a.s. then

$$corr_t\left(Z_\tau, Y_\tau\right)^2 \geq corr_t\left(Z_\tau, Y_\sigma\right)^2.$$

Hence, we choose to sample controls at the exercise time, that is

$$Y\left(\tau\right) = E_\tau^{\mathbb{Q}}\left(Y_T\right),\ \tau \leq T.$$

5 Bermudan Swaptions in Libor Market Models

5.1 Libor Market Models

Since the seminal papers of Miltersen, Sandmann & Sondermann (1997), Brace & Musiela (1997) and Jamshidian (1997), Libor market models have become increasingly popular in the practitioners' world, as the models are reasonably easy to calibrate and allow closed form solutions for caps and swaptions (though not simultaneously). Several extensions have been proposed. Jamshidian (1999) develops a general theory for Libor market models driven by semimartingales. Along this line Glasserman & Kou (1999) developes a version with jumps, driven by a marked point process, which has closed form solutions for certain derivatives. We follow the approach taken by Andersen & Brotherton-Ratcliffe (2001) who developed an extended Libor Market model with a continuous stochastic volatility process independent of the forward rates themselves.

The Libor market model is defined on an increasing tenor structure $0 = T_0 < T_1 < ... < T_{K+1}$. Let $\eta(t)$ denote the right continuous mapping function returning the index of the next tenor time $\eta(t) = \{j : T_{j-1} < t \leq T_j\}$. The simple forward rate $F_k\left(t\right)$ for the period T_k to T_{k+1} is defined by

$$F_k\left(t\right) = \frac{1}{\delta_k}\left(\frac{P\left(t, T_k\right)}{P\left(t, T_{k+1}\right)} - 1\right),\ \delta_k = T_{k+1} - T_k,$$

where $P\left(t, \cdot\right)$ denotes the time t discount function. We work under the "spot Libor" measure $\mathbb{Q}$ (see e.g. Jamshidian (1997)) under which all assets discounted by the "Bank" account $B\left(t\right)$ are martingales. $B\left(t\right)$ is the value of an initial \$1 investment, rolled over at the spot Libor rate at each tenor date

$$B(t) = P\left(t, T_{\eta(t)}\right)\prod_{k=0}^{\eta(t)-1}\left(1 + \delta_k F_k(T_k)\right).$$

Andersen & Brotherton-Ratcliffe (2001) model stochastic volatility with a variance process $V(t)$ used to scale the diffusion term of all forward rates $\varphi(F_k(t))\lambda_k(t)$, $k = 1, .., K$.

Using no-arbitrage arguments (see e.g. Jamshidian (1999)), it can be proven that the full dynamics under the spot Libor measure $\mathbb{Q}$ of the $(K+1)$ −Markov system of forward Libor rates, is given by

$$dF_k(t) = \varphi(F_k(t))\sqrt{V(t)}\lambda_k(t)\left(\mu_k(t)\,dt + dW_t^{\mathbb{Q}}\right), \; k = 1, .., K \tag{10}$$

$$dV(t) = \kappa_V(\theta_V - V(t))\,dt + \varepsilon_V\kappa_V(\theta_V - V(t))\,dt + \varepsilon_V\psi(V(t))\,dZ_t^{\mathbb{Q}} \tag{11}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a one dimensional function satisfying certain regularity conditions, $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is well-behaved, $\lambda_k(t)$ is a bounded deterministic function taking values in $\mathbb{R}^{1\times m}$, $\kappa_V, \theta_V, \varepsilon_V$ are positive constants and $W^{\mathbb{Q}}$ and $Z^{\mathbb{Q}}$ are m-dimensional and one dimensional Brownian motions under $\mathbb{Q}$, respectively. The drift function for the k-th Libor rate is given by

$$\mu_k(t) = \sum_{j=\eta(t)}^{k} \frac{\delta_j\varphi(F_j(t))}{1+\delta_j F_j(t)}\lambda_j(t)^T. \tag{12}$$

For the approximation formulae for European options derived by Andersen & Brotherton-Ratcliffe (2001) to hold, it is important that the Wiener processes driving the variance process and the Libor rates are uncorrelated. We do not consider calibration issues in this paper but keep the assumption anyway[2]. Furthermore, it is natural to scale the variance process such that $\theta = 1$, meaning that $V(t) - 1$ represents a percentage deviation from the long term mean of the variance.

5.2 Exercise strategies

An exercise strategy is basically all that we need to get the Primal-Dual algorithm running. Several possibilities exist regarding an appropriate exercise strategy, including exercise strategies from lower dimensional models which can be solved using lattice methods, least-square Monte Carlo (LS) (see Longstaff & Schwartz (2001)), neural networks, or trigger strategies like the ones explored in Andersen (2000). We have chosen the latter, because of its simplicity and speed. Bjerregaard Pedersen (1999) finds the Longstaff & Schwartz (2001) and Andersen (2000) approaches to be mutually consistent for Bermudan swaptions.

The idea in trigger strategies is to reduce the dimensionality of the exercise decision. We consider exercise strategies of the following form. First, we let $X(t_i)$ denote the option payoff if exercised at time t_i. The exercise rules specify

[2] This is also in agreement with theory and empirical evidence regarding unspanned stochastic volatility documented in Collin-Dufresne & Goldstein (2002). Nonzero correlation between volatility variance and Libor rates would (in principle) make it possible to hedge all assets using only bonds.

exercise if the payoff is larger than some parameterized function f of the state vector $Z(t_i)$.

$$I(t_i) = \begin{cases} 1 & \text{if } X(t_i) > f(Z(t_i); \mathbf{p}(t_i)) \\ 0 & \text{otherwise} \end{cases} .$$

Among the exercise strategies we consider for Bermudan swaptions are the ones proposed in Andersen (2000). First, we let $M(t_i) = \max_{j=i+1,\dots,e} (EO_{j,e}(t_i))$ denote the most valuable of the still alive core European options $EO_{j,e}(t_i)$.

Strategy 1:

$$I^1(t_i) = \begin{cases} 1 & \text{if } X(t_i) > H_1(t_i) \\ 0 & \text{otherwise} \end{cases}$$

Strategy 2:

$$I^2(t_i) = \begin{cases} 1 & \text{if } X(t_i) > H_2(t_i) \ \textit{and} \ X(t_i) > M(t_i) \\ 0 & \text{otherwise} \end{cases}$$

Strategy 3:

$$I^3(t_i) = \begin{cases} 1 & \text{if } X(t_i) > H_3(t_i) + M(t_i) \\ 0 & \text{otherwise} \end{cases}$$

Strategy 4:

$$I^4(t_i) = \begin{cases} 1 & \text{if } X(t_i) > H_2(t_i) \ \textit{and} \ X(t_i) > EO_{t_{i+1},e}(t_i) \\ 0 & \text{otherwise} \end{cases}$$

Strategy 5:

$$I^5(t_i) = \begin{cases} 1 & \text{if } X(t_i) > H_3(t_i) + EO_{t_{i+1},e}(t_i) \\ 0 & \text{otherwise} \end{cases}$$

Here $H_i(t)$ denotes a deterministic function from $R^+ \to R$. In *Strategy 2*, the payoff should be larger than the barrier and the maximal value of the remaining European options. Finally, in *Strategy 3*, the payoff should exceed the sum of the barrier and the maximal value of remaining European options. Andersen (2000) finds that in certain scenarios the more advanced strategies pick up additional value compared to the first strategy, but they also increase the computational burden - especially for options with several exercise times. When computing the maximal value of the remaining European options required for *Strategy 3*, the worst case number of option valuations for each path is of order n^2, where n is the number of exercise times. In worst case, an option with 40 exercise times would require as much as $40(40+1)/2 = 820$ European option valuations for each path. The rationale for these strategies is of course that the holder of the American option could always sell it to the value of the most valuable European option.

We propose a small simplification of the strategies using the maximal of the European values, namely to use only a subset of the core European options as these are likely to be correlated. In the Bermudan swaption case considered below, the most valuable core swaption is almost always the first to mature (the underlying swap is the longest). So, we test two simplifications of the strategies 2 and 3, in which we replace the maximal of the remaining swaptions with the first to mature European swaption $ES(t_i)$. These are denoted *Strategy* 4 and *Strategy* 5, respectively, and we return to their performances in section 6.4.

The critical barriers $H_i(t)$ are found using an optimization procedure described in Andersen (2000), consisting of a presimulation of paths and a series of one-dimensional optimizations at each exercise time.

5.3 Bermudan Swaptions

The *Bermudan swaption* (payer) is basically an option to enter into a (payer) swap contract with a coupon of κ. The standard product denoted $BS_{s,x,e}$ is characterized by three dates: *the lockout date* T_s, the *last exercise date* T_x and the *final swap maturity* T_e. We will consider the fixed maturity case where $T_s < T_x < T_e$, allowing early exercise on dates in the set $\mathcal{T} = \{T_s, T_{s+1}, \ldots, T_x\}$. If the option is exercised at time T_k, the holder of the option receives the payoff from the corresponding European swaption with $e-k$ periods and coupon κ. If exercised at time $T_k \in \mathcal{T}$ the payoff is

$$\begin{aligned} BS_{s,x,e}(T_k,\kappa) &= (Swap(T_k,T_k,T_e,\kappa))^+ \\ &= \left(1 - P(T_k,T_e) - \kappa \sum_{i=k}^{e-1} \delta_i P(T_k,T_{i+1})\right)^+ . \end{aligned}$$

From (1) we know that the value at time $t \le T_{e-1}$ of the Bermudan payer swaption BS is given as the solution to the optimal stopping problem

$$BS_{s,x,e}(t,\kappa) = \sup_{\tau \in \Gamma(t)} E_t^{\mathbb{Q}} \left[\frac{B(t)}{B(\tau)} (Swap(\tau,\tau,T_e,\kappa))^+ \right]. \tag{13}$$

In this paper we only consider Bermudan swaptions where $x = e-1$. A Bermudan swaption with lock-out date T_s and final swap maturity T_e will be denoted T_e no-call T_s, T_e nc T_s, or T_s into T_e.

5.4 Possible control variates

In this section we consider several assets which we test as control variates. However, it is important to notice that when applying the Rasmussen (2002) technique of sampling at the exercise time, we have to be careful with assets that have several payoff times, as the ex-dividend value of an asset is a supermartingale. If a control asset $X(t)$ is generating a dividend process $\{\gamma(T_k)\}_{1 \le k \le K+1}$, we know that the time t value of the remaining discounted dividends is a martingale and given by

$$\begin{aligned}
\frac{X(t)}{B(t)} &= E_t^{\mathbb{Q}}\left[\sum_{i=\eta(t)}^{K+1}\frac{\gamma(T_i)}{B(T_i)}\right] \\
&= E_t^{\mathbb{Q}}\left[E_\tau^{\mathbb{Q}}\left(\sum_{i=\eta(t)}^{K+1}\frac{\gamma(T_i)}{B(T_i)}\right)\right] \\
&= E_t^{\mathbb{Q}}\left[\sum_{i=1}^{\tau}\frac{\gamma(T_i)}{B(T_i)} + E_\tau^{\mathbb{Q}}\left(\sum_{i=\tau+1}^{K+1}\frac{\gamma(T_i)}{B(T_i)}\right)\right] \\
&= E_t^{\mathbb{Q}}\left[\sum_{i=1}^{\tau}\frac{\gamma(T_i)}{B(T_i)} + \frac{X(T_\tau)}{B(T_\tau)}\right], \qquad (14)
\end{aligned}$$

for a given stopping time $\tau \in \Gamma(t)$. Here we have used the *Law of Iterated Expectations* for the second equality.

Equation (14) states that, as the initial value includes all future dividends received from the asset, we have to include all dividends reinvested in the numeraire asset[3] in the sampling value Y of the asset X. Hence we sample

$$Y(\tau) = \sum_{k=1}^{\tau}\frac{\gamma(T_k)}{B(T_k)} + \frac{X(T_\tau)}{B(T_\tau)}. \qquad (15)$$

Notice that assets maturing before the option expires can still be used (e.g. zero coupon bonds).

5.4.1 European Swaptions

From (13) an obvious choice of controls would be to use the discounted payoff from the core European swaptions $ES_{k,e}(\cdot,\kappa)/B(\cdot)$ for $T_k = T_s, T_{s+1}, \ldots, T_x$ which are $\mathbb{Q}$ martingales[4]. However, only approximative closed form solutions for swaptions have been derived (see e.g. Brace & Musiela (1997) or Andersen & Andreasen (2000)), which in effect means that European swaptions are not applicable as controls in a Libor market model. Another possibility is to create a self-financing portfolio replicating as closely as possible the European swaptions by using the hedge ratios from the approximative solutions, as any discounted self-financing strategy has to be a martingale in a no-arbitrage setting. This approach is described below in section 5.4.4.

[3] All assets give the same return under the spot Libor measure. Hence, anyone can be used.

[4] We cannot use the discounted forward swap rate as a control as this is not a martingale under the measure $\mathbb{Q}$. Each forward swap rate $\omega_{T_k}(\cdot,e)$ is a martingale under the forward swap measure $\mathbb{Q}^{k,e}$ induced by the numeraire $B^{k,e}(\cdot)$.

5.4.2 Caps and Caplets

A more simple solution is to consider caps and caplets, as these have closed form solutions in a standard Libor market model (see e.g. Brace & Musiela (1997) or Andersen & Andreasen (2000)). We note that a caplet is a one-period swaption and hence the price of the cap will always be higher than that of a swaption with the same coupon, start date and end date, and thus make an upper bound of the price of the swaption. The question is now which caps or caplets should be used as controls. In theory, including all possible control variates will never decrease efficiency, but in practice we could experience problems with multicollinearity.

Hence, a first choice of including all caplets with starting dates in $\mathcal{T}(t)$ with a strike equal to the coupon of the Bermudan swaption would mean sampling all $x-s$ controls[5]. If we let $\wedge$ denote the minimum operator the sampled values are

$$Y_k(\tau) = \frac{Caplet\,(\tau \wedge T_k, T_k, \kappa)}{B(\tau)}.$$

We have just argued that a cap with same strike rate and start- and maturity date as a Bermudan swaption, only constitutes an upper bound, but could be a reasonable control. In principle it is just a portfolio of caplets and one could suspect that we might gain more by including all constituent caplets as separate controls. This is generally not the case as we avoid a lot of multicollinearity using the cap. By using the cap we essentially estimate a common β for all caplets. Referring to (15), the sampling value $Y(\tau)$ is

$$Y(\tau) = \sum_{k=1}^{\tau} \frac{(F_{k-1}(T_{k-1}) - \kappa)^+}{B(T_k)} + \frac{Cap(T_\tau, T_{e-1}, \kappa)}{B(T_\tau)}.$$

5.4.3 Swaps and Zero Coupons

Standard fixed income securities can be used as control variates in the Libor market models as the yield curve is known. We consider using the swap underlying the Bermudan as control variate. Thus we sample

$$Y(\tau) = \sum_{k=s}^{\tau} P(T_k, T_{k+1}) \frac{\delta_k (F_k(T_k) - \kappa)}{B(T_k)} + \frac{Swap\,(\tau, \tau+1, T_e, \kappa)}{B(\tau)}.$$

We also test a series of zero coupon bonds with maturity dates equal to the Bermudan option's set of exercise dates $\mathcal{T}$ as controls.

$$Y_k(\tau) = \frac{P(\tau \wedge T_k, T_k)}{B(\tau \wedge T_k)}, \; k \in \mathcal{T}.$$

Notice that the swap is a portfolio of zero coupon bonds and, hence, the difference between the two control types is basically that we restrict the β parameter when using a swap as control.

[5] For upper bound calculations we will use ATM caplets. Thus, when calculating $E_t(\cdot)$ we fix the strike at time t such that the caplet is ATM.

5.4.4 Approximate Delta-Hedge

The value of any self-financing portfolio discounted with the pricing numeraire must be a martingale. Hence, we know that the expected value of such a portfolio is the initial value of the portfolio. As such we can use the payoff from a portfolio that approximately replicates the option we are trying to value. (see e.g. Clewlow & Carverhill (1994) for more simple examples).

As a quite general control variate we test a hedge portfolio. More specifically, we form a portfolio delta hedging the first of the remaining core European swaptions to expire using the delta implied by the approximate swaption formula given in Andersen & Andreasen (1998). Hence,

$$\Delta_{s,e}(t) = \frac{\partial ES_{s,e}(t,\kappa)}{\partial S_{s,e}(t)} = B_{s,e}(t)\Phi\left(d_+\right)$$

where we denote the *forward swap rate* and the *accrual factor* by

$$S_{s,e}(t) = \frac{P\left(t,T_s\right) - P\left(t,T_e\right)}{\sum_{k=s+1}^{e} \delta_{k-1} P\left(t,T_k\right)}$$

and

$$B_{s,e}(t) = \sum_{k=s+1}^{e} \delta_{k-1} P\left(t,T_k\right),$$

respectively. Φ denotes the standard normal cumulative density function and

$$d_+\left(t,T_s\right) = \frac{\log \frac{\omega_{T_s}(t,e-s)}{\kappa} + \frac{1}{2}\xi^2\left(t,T_s\right)}{\xi\left(t,T_s\right)}$$

where $\xi\left(t,T_s\right)$ is the approximative integrated volatility. That is, at time t_j we buy

$$\begin{array}{lcc} \Delta_{s,e}(t_j)\cdot S_{s,e}\left(t_j\right) & , & t_j < T_s \\ \Delta_{j+1,e}(t_j)\cdot S_{j+1,e}\left(t_j\right) & , & T_s \le t_j \le T_e \end{array}$$

financing it with a short position in the bank account. At the following tenor date we liquidate the swap position and enter a new, putting the profits or losses in the bank account. This is done until exercise or maturity.

The drawback of this general approach is that it requires the hedge portfolio to be updated all along each path, which is computationally expensive. Furthermore, as the hedging errors are decreasing in the number of resettings, and we only reset at the tenor dates, we do not expect a correlation of the payoff very close to 1. Numerical tests will show how well this strategy performs.

A more precise hedging strategy would be to take into account the model dynamics of the swap rate and try to hedge each factor driving the yield curve. Though, increasing precision should be measured against increases in computation time.

6 Numerical Results

Libor market models cannot be simulated exactly. However, the resulting discretization biases are manageable and small compared to the variance, as shown in e.g. Andersen & Andreasen (1998), and are ignored in the following analysis. The exact simulation schemes have been included in the appendix for completeness.

6.1 Benchmarking

To be able to compare with the results found in Andersen & Broadie (2001) we use one of their test scenarios. We start out with a standard log-normal Libor market model with a quarterly tenor structure and deterministic volatility corresponding to letting $\varphi(x) = x$, $\varepsilon = 0$ and $V(0) = 1$ in equation (10). The initial yield curve is flat 10% and we investigate a two-factor version with the following deterministic volatility structure.

$$\lambda_k(t) = (0.15\ ,\ 0.15 - \sqrt{0.009 \cdot (T_k - t)})^{\intercal}$$

To benchmark our model implementation we include Table 1 which replicates a part of Table 4 presented in Andersen & Broadie (2001).

Table 1: Benchmark Scenario

$T_{\mathbf{s}}$	$T_{\mathbf{e}}$	*Strike*	*Lower*	*Std Low*	$\hat{D}_{\mathbf{0}}$	*Std* $\hat{D}_{\mathbf{0}}$	*Time*
1.00	3.00	0.08	339.41	0.24	0.34	0.05	00:01:32
1.00	3.00	0.10	124.82	0.34	0.55	0.07	00:02:29
1.00	3.00	0.12	35.89	0.24	0.44	0.07	00:03:07
1.00	6.00	0.08	749.59	0.55	3.09	0.26	00:11:20
1.00	6.00	0.10	317.10	0.68	4.75	0.32	00:20:29
1.00	6.00	0.12	126.29	0.60	2.52	0.26	00:26:38
1.00	11.00	0.08	1249.53	1.24	18.87	1.31	01:10:11
1.00	11.00	0.10	620.62	1.19	19.99	1.09	01:57:07
1.00	11.00	0.12	329.89	1.17	14.11	0.97	02:27:57

Benchmarking of the lower bound estimates and the duality gaps. T_s denotes the lock-out period of the Bermudan swaption and T_e is the final swap maturity. Strike denotes the coupon of the underlying swap. Lower and Std Low are the lower bound point estimate and it's standard error found using 50.000 antithetic paths and strategy 1. D_0 is the point estimate of the duality gap and $StdD_0$ it's standard deviation based on $n = 750$ antithetic paths and $m = 300$ antithetic paths used for "simulation within the simulation". Time denotes the computation time for the duality gap in hours:minutes:seconds.

Table 1 contains estimates of the lower bound and the duality gap for a set of Bermudan swaptions. Even though we have estimated our own exercise barriers we cannot distinguish the estimates from the ones presented in Andersen &

Broadie (2001) as the confidence intervals are overlapping. This goes for both the lower and upper bounds.

6.2 Efficiency Improvement

6.2.1 Caps and caplets as control variates

Figure 1 illustrates the effect of the number of caplets used as control variates on the standard deviation. In this setup we have sampled the first and the last possible control and sampled remaining controls equally spaced. We see that for the 11 nc 1 payer Bermudan swaption, about seven caplet controls, equally spaced over the exercise period, should be sufficient and that sampling all 40 caplets will not improve the estimation result significantly.

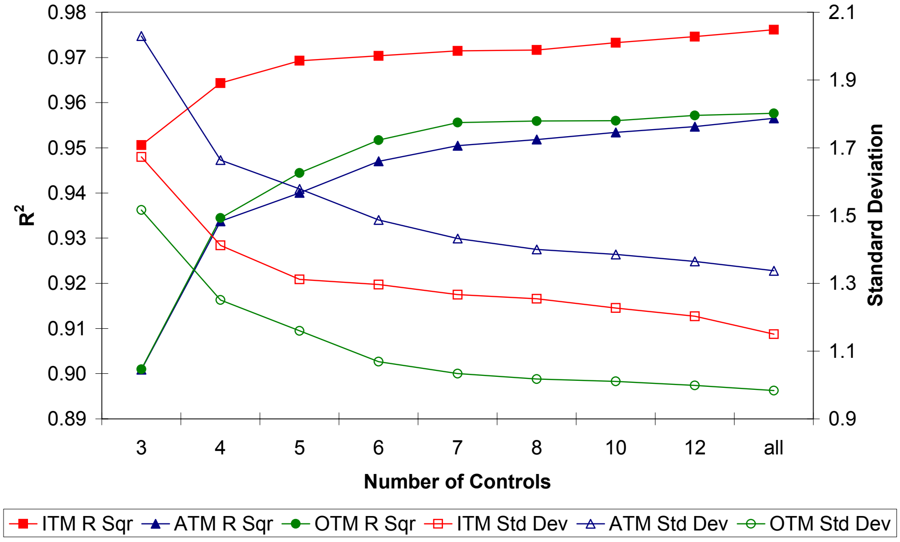

Figure 1: Illustration of the effect of the number of caplet control variates on R^2 and sample standard deviation for an ITM, ATM and OTM 1-11 Bermudan payer swaption. 2-dimensional Libor Market Model with deterministic volatility and using 10,000 simulation paths.

If we were only interested in reducing the standard deviation of the lower bound estimates, we would include all caplets. However, as we show in the following section, we also have to take computation time into consideration. Furthermore, we have to balance the total number controls relative to the number of paths to avoid multicollinearity as already mentioned in section 5.4.2.

6.2.2 Testing controls variates

Comparing the efficiency of variance reduction techniques is always difficult, as they depend on the implementation. However, it is still relevant to compare *particular* implementations. In Glynn & Whitt (1992) it is proved that if we compare two estimators with variances σ_1^2 and σ_2^2 requiring an *expected* work per run of Ew_1 and Ew_2, respectively, we should favor the estimator with the lowest variance scaled with the workload $\sigma_i^2 \cdot Ew_i$. We denote this product the efficiency factor EF_i.

In Table 2 we test the efficiency on lower bound estimates of different types of control variates on 6 nc 1 and 11 nc 1 Bermudan swaptions of various strikes. The table shows the standard deviations and the variances (in basis points) scaled with the average time per path. We have tested two caplet setups: the case of 7 caplets and all caplets. As we pointed out in section 6.2.1, using all caplets might not always be the best choice for cases of long maturity options.

For both the 6 nc 1 and 11 nc 1 Bermudan swaption we see that both the cap and the series of caplets perform almost equally well for all strikes, the series of caplets doing slightly better than the cap for the OTM options. Especially, we note that the more the option is in-the-money, the less efficient it is to use all caplets compared to only using 7. However, the differences in the efficiency factor between the caplet types and the cap control variates are negligible compared to the other types of controls which perform quite poorly in comparison.

We note that using the cap compared to a series of caplets is easier to apply in the sense that we do not have to decide how many caplets to use - including too many or too few caplets may cause the caplet control to be less efficient than the single cap.

From Table 2 we see that using the self-financing delta hedge strategy results in a reasonably low standard deviation - yet still not better than the caplets or single cap - but when taking the computational effort into account it does not constitute a good control variate. Especially not for out of the money options.

The variance reduction obtained by using the zero coupon bonds and the swaps is of similar size in all cases. But as the swap uses more computer time, it is better to use the zero coupon bonds.

As usual the antithetic sampling works better for in the money options as the payoff function is close to being linear. In Table 3 we present results using antithetic variates and control variates. We have used 25,000 antithetic paths - i.e. a total of 50,000 paths - in order to be able to compare table 2 and table 3. The results are similar to the previous results. The single cap and the caplets still outperform the other types of controls, and the zero coupon bonds are more efficient than the swaps.

Comparing table 2 with table 3 we notice that the standard deviations and efficiency factors given in table 3 are lower than those of table 2 except for the case of the delta hedge control variate.

We also test the effect of using multiple controls simultaneously. As before we run tests in a two-factor Libor model using *Strategy 1* and 50.000 paths

Table 2: Efficiency of control variates

Crude MC		*6 nc 1*		*11 nc 1*	
Coupon	*CV*	*StdLow*	*EF*	*StdLow*	*EF*
0.08	Antithetic	0.5582	0.0804	1.1291	0.8570
0.08	Cap	0.1856	0.0061	0.5182	0.1115
0.08	Caplets	0.1843	0.0057	0.5357	0.1072
0.08	Caplets All	0.1761	0.0063	0.5046	0.1267
0.08	DeltaHedge	0.3240	0.0319	0.7485	0.4055
0.08	None	2.0469	0.6218	3.2819	3.7633
0.08	Swap	0.5254	0.0525	1.1693	0.6760
0.08	Z-Bond	0.5259	0.0453	1.1599	0.5527
0.10	Antithetic	0.6781	0.2146	1.1372	1.7359
0.10	Cap	0.2620	0.0186	0.6314	0.3013
0.10	Caplets	0.2675	0.0191	0.6153	0.2740
0.10	Caplets All	0.2522	0.0189	0.5728	0.2639
0.10	DeltaHedge	0.4982	0.1257	1.0507	1.4600
0.10	None	1.5379	0.5974	2.7992	5.3533
0.10	Swap	0.7379	0.1627	1.4140	1.6782
0.10	Z-Bond	0.7368	0.1467	1.4164	1.4921
0.12	Antithetic	0.6005	0.2111	1.1716	2.7579
0.12	Cap	0.2001	0.0130	0.5102	0.2314
0.12	Caplets	0.1977	0.0128	0.4875	0.2078
0.12	Caplets All	0.1858	0.0121	0.4382	0.1890
0.12	DeltaHedge	0.5185	0.1695	1.2019	2.3707
0.12	None	1.0172	0.3274	2.1852	4.1335
0.12	Swap	0.6499	0.1531	1.3210	1.7828
0.12	Z-Bond	0.6418	0.1371	1.3167	1.6017

This table contains a summary of efficiency rates for the control variates considered in this paper applied on the 6 nc 1 and 11 nc 1 Bermudan swaptions in the two-factor deterministic volatility scenario. Coupon is the coupon rate of the underlying swap. CV is the control variate and std Low denotes the standard deviation of the lower bound using strategy 1. EF denotes the variance in (bps) scaled with the average time per path. Low values of EF are preferred. The numbers were generated using 50,000 paths.

Table 3: Efficiency of Control Variates with Antithetic Sampling.

Antithetic		6 *nc* 1		11 *nc* 1	
Coupon	*CV*	*StdLow*	*EF*	*StdLow*	*EF*
0.08	Antithetic	0.7910	0.0826	1.6126	0.8883
0.08	Cap	0.1755	0.0050	0.5101	0.1053
0.08	Caplets	0.1544	0.0035	0.4946	0.0874
0.08	Caplets All	0.1441	0.0038	0.4664	0.1001
0.08	DeltaHedge	0.3823	0.0427	0.8422	0.5088
0.08	Swap	0.3681	0.0230	0.9041	0.3736
0.08	Z-Bond	0.3637	0.0188	0.9097	0.3091
0.10	Antithetic	0.9576	0.2166	1.6121	1.7755
0.10	Cap	0.2028	0.0105	0.5427	0.2112
0.10	Caplets	0.2298	0.0131	0.5861	0.2381
0.10	Caplets All	0.2043	0.0115	0.5129	0.2072
0.10	DeltaHedge	0.5256	0.1350	0.9195	1.1137
0.10	Swap	0.7033	0.1363	1.3863	1.5601
0.10	Z-Bond	0.7032	0.1218	1.3981	1.4042
0.12	Antithetic	0.8456	0.2121	1.6562	2.3656
0.12	Cap	0.1892	0.0109	0.4684	0.1974
0.12	Caplets	0.1853	0.0105	0.4489	0.1762
0.12	Caplets All	0.1685	0.0092	0.3979	0.1529
0.12	DeltaHedge	0.4017	0.0968	0.7565	0.9380
0.12	Swap	0.6183	0.1281	1.3616	1.8555
0.12	Z-Bond	0.6182	0.1176	1.3447	1.6409

This table contains a summary of efficiency rates for the control variates considered in this paper applied on the 6 nc 1 and 11 nc 1 Bermudan swaptions in the two-factor deterministic volatility scenario. Coupon is the coupon rate of the underlying swap. CV is the control variate and std Low denotes the standard deviation of the lower bound using strategy 1. EF denotes the efficiency factor i.e. the variance in (bps) scaled with the average time per path. Low values of EF are preferred to high. The numbers were generated using 25,000 (AS) paths.

(Crude MC). In Table 4 we have listed the 15 most efficient types of control variates for a 11 nc 1 Bermudan with varying strikes.

We see that the caplet control variate still performs reasonably well compared to using multiple control types, especially for out of-the-money options. However, using a series of caplets or single cap together with the zero coupon bond or swap, is more efficient than using only the caplets; except for the out of-the-money Bermudan swaption, where it is more efficient to use only the caplet instead of both the caplet and the swap. As expected the zero coupon bond, together with the cap/caplets, are more efficient than the swap together with the cap/caplets. The swap, cap and zero coupon also perform well for in- and at-the-money Bermudans. Finally we point out that the delta hedge control is far too time consuming compared to the other types of controls.

Table 4: Efficiency of combinations of control variates.

	Coupon					
	0.08		0.10		0.12	
Rank	*EF*	*CV*	*EF*	*CV*	*EF*	*CV*
1	0.0767	ZB+Cap	0.2173	ZB+Cap	0.1821	ZB+Cap
2	0.0833	ZB+CL	0.2378	SW+Cap	0.1885	ZB+CLA
3	0.0898	ZB+SW+Cap	0.2428	ZB+CL	0.1890	CLA
4	0.0902	SW+Cap	0.2523	ZB+SW+Cap	0.1914	ZB+CL
5	0.0962	SW+CL	0.2536	ZB+CLA	0.2033	SW+CLA
6	0.1038	ZB+CLA	0.2639	CLA	0.2078	CL
7	0.1072	CL	0.2678	SW+CL	0.2083	SW+Cap
8	0.1092	SW+CLA	0.2740	CL	0.2116	SW+CL
9	0.1115	Cap	0.2797	SW+CLA	0.2314	Cap
10	0.1169	ZB+SW+CL	0.3013	Cap	0.2443	ZB+SW+Cap
11	0.1267	CLA	0.3336	ZB+Cap+DH	0.2641	CL+DH
12	0.1334	ZB+Cap+DH	0.3339	ZB+SW+CLA	0.2715	ZB+CLA+DH
13	0.1379	ZB+CL+DH	0.3405	ZB+SW+CL	0.2756	ZB+SW+CL
14	0.1396	SW+Cap+DH	0.3599	SW+Cap+DH	0.2838	CLA+DH
15	0.1464	ZB+CLA+DH	0.3633	ZB+CLA+DH	0.2840	SW+CLA+DH

This table contains the results of test runs using various control variates simultaneously when pricing a 11 no-call 1 Bermudan swaption for various coupons of the underlying. To ease the exposition we have ranked the combinations based on their performance. EF denotes the efficiency factor i.e. the product of variance and average time per path. CV columns contain the combinations of control variates. We have used the following abbriviations: ZB=Zero Coupon, SW=Swap, CL=Subsample of Caplets, CLA=Caplets All and DH=Delta Hedge.

6.3 The Bias of the Upper Bound

Using Jensen's inequality it is easily shown that the estimator for the duality gap is upward biased. Andersen & Broadie (2001) do not examine the size and

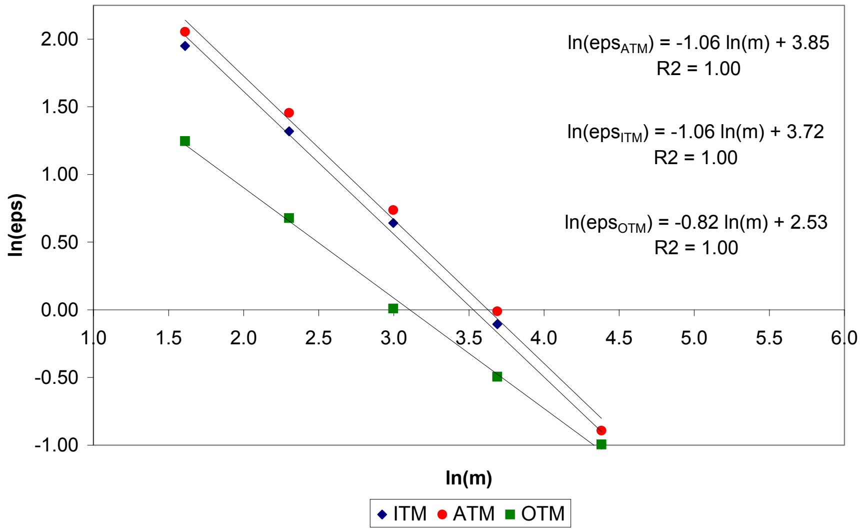

Figure 2: The graph shows the results from our estimation of the rate β at which the bias decrease. eps denotes the estimate of the bias and m the number of inner simulation paths. These results are for the 6-11 years Bermudan swaption for variuos coupon rates of the underlying swap in the two-factor deterministic volatility scenario. The strategy used was strategy 1. It also contains simple OLS-regression results for minus the β parameters. For practical applications a $\beta = 1$ seems to be a reasonable value.

behavior of the upper bound. This is particularly interesting as we want to make an efficient choice of the number of inner m and outer paths n, which is basically a variance-bias trade-off where increases in m decreases bias ε and increasing n decreases variance. We balance these using the mean square error MSE given by

$$MSE = \varepsilon^2 + Var\left(\hat{D}\right).$$

In Appendix B we propose an allocation rule of the form that states how to increase the number of inner paths as we increase outer paths

$$m(n) \propto n^{\frac{1}{2\beta}},$$

where β is the rate of which the bias decreases. We also propose a procedure that gives an estimate of β. Basically we run a regression corresponding to equation (16) in appendix. The output of this procedure is shown in figure 2.

We have tested this method using strategy 1 in the two factor deterministic volatility scenario using $n = 10,000$ antithetic paths and $m = 5, 10, 20, 40$ and 80 in the regression. For these values of m, the size of the bias is still large relative to the standard deviation. The estimate of the true D is based on $n^* = 10,000$ and $m^* = 1,280$. Figure 2 also shows that the β estimates are close to 1. This means that when we double the number of outer paths we should only increase the number of inner paths with a factor of $\sqrt{2}$.

6.4 Testing Strategies

So far we have only used the simple exercise strategy 1. To test the implications of this we now take a further look at the more complicated strategies 2-5. When using these strategies we have to estimate the value of the remaining core European swaptions at each possible exercise time. This is done using an approximative European swaption formula found in Andersen & Andreasen (1998) (see e.g. Brace & Musiela (1997)).

Andersen (2000) demonstrated that strategy 1 works well for most Bermudan swaptions across several scenarios of the economy. Short options on long swaps in the multifactor model, were the only cases where the more complicated strategies really picked up additional value. Our tests show the same and for that reason we concentrate on the 11 nc 1 Bermudan swaption in the two-factor Libor market model with deterministic volatility. Computations are based on 50,000 AV paths and to reduce sample standard deviation we use the forward starting cap, covering the same period as the swaption, as a control variate.

Several interesting observations can be made from Table 5. First of all, the enhanced strategies do pick up additional value with a maximum of 12 basis points for the at-the-money swaption. Secondly, strategy 3 results in increased lower bounds compared to strategy 2, but is also slower to compute. This is due to the fact that all core European swaptions have to be computed in order to determine whether to exercise or not, whereas calculations of the core swaptions in strategy 2 should be skipped as soon as one is more valuable than intrinsic value. This effect is very pronounced in the out of-the-money case where calculation times are significantly different.

One possible explanation as to *why* strategy 3 picks up additional value could be that the barrier $H_3(t_i)$ in strategy 3 could be interpreted as the value of the deferred exercise premium at the exercise boundary. Remember that we are searching for the $H_3(t_i)$ that would make intrinsic value equal to the continuation value, $X(t_i) = H_3(t_i) + M(t_i)$. In strategies 1 and 2 the barrier should approximate the sum $H(t_i) + M(t_i)$ instead of just part of it. Furthermore, since the early/deferred exercise value goes to zero when the option is deep in- or out of-the-money this might be easier to approximate. This could indicate that in cases where we have good approximations for European core options, the parametrization of exercise strategies should be formulated in the early/deferred exercise values rather than just testing on the European value as in strategy 2.

Strategies 4 and 5 result in prices which are not significantly lower than those implied by strategies 2 and 3, respectively, confirming our initial expectations. However, computational savings are significant, especially for the options with long exercise periods which are also the options that would benefit from

Table 5: Performance of strategies 1-5

Coupon	*Strategy*	*Low* *bps*	*Pickup* *bps*	*Time* *min : sec*	*Relative* *Time*
0.08	1	1248.3 (0.4)		00:59	1.0
0.08	2	1251.0 (0.4)	2.7	19:01	19.3
0.08	3	1255.5 (0.4)	7.2	29:26	29.9
0.08	4	1251.8 (0.4)	3.5	01:11	1.2
0.08	5	1254.9 (0.3)	6.6	01:13	1.2
0.10	1	622.5 (0.4)		01:43	1.0
0.10	2	628.9 (0.4)	6.4	16:37	9.7
0.10	3	635.1 (0.3)	12.6	35:14	20.5
0.10	4	628.7 (0.4)	6.2	01:59	1.2
0.10	5	635.4 (0.3)	12.9	01:59	1.2
0.12	1	329.8 (0.3)		02:05	1.0
0.12	2	334.4 (0.3)	4.6	07:42	3.7
0.12	3	338.2 (0.3)	8.4	21:52	10.5
0.12	4	334.9 (0.3)	5.1	02:15	1.1
0.12	5	338.9 (0.3)	9.1	02:17	1.1

This table summarize the results of a test run of exercise strategies 1 to 5. It contains lower bound estimates, standard deviation and main simulation times for the 11 no call 1 Bermudan swaptions for various coupons. 50,000 antithetic paths and the cap as control variate was used in the simulation. Time is presented as minutes:seconds. The pickup from using the advanced strategies relative to the simple strategy 1 is included as well as the computation time relative to strategy 1. The strategies where estimated using 50,000 antithetic paths.

enhancement of the simple strategy. These savings do not only stem from the lower number of swaptions, but also the particular choice of swaptions. Remember that short maturity swaptions are much cheaper to evaluate than long maturity swaptions as we need to integrate the volatility to the expiry of the option.

Strategy 4 and strategy 5 are only about 20% slower than strategy 1 and much faster than strategies 2 and 3. The computation times for strategy 4 range from 6% to 29% of strategy 2 and as little as 4% to 10% for strategy 5 compared to strategy 3. These numbers would be even lower if we included computation time used in the presimulation.

The conclusion is quite clear. For Bermudan swaptions with many exercise times strategy 5 is preferred, as the additional computational cost is low while the lower bounds are as high as the ones from strategy 3.

6.5 Price sensitivity to the estimated barrier

We have tested the sensitivity of the exercise barrier on the upper and lower bound estimates. This is done by scaling the exercise barrier by a constant α and then computing lower and upper bound estimates. Noting here that scaling the barrier only changes the level of the barrier not the shape. The results are plotted in figure 3 for various values of α. Interestingly, the lower bound seems to be more sensitive to the barrier scaling factor than the upper bound.

Again, tests have been performed in the two factor deterministic volatility scenario using exercise strategy 1. The Bermudan swaption is a 1 into 6 year with a coupon of 10%.

6.6 Upper Bound Calculations

We are now ready to compute the duality gaps and upper bound estimates. However, as our computational results are virtually identical to the findings in Andersen & Broadie (2001), we will mainly focus on the simplified strategies 4 and 5 as well as the effect of control variates on the estimation of the upper bound. There is, however, only minor discrepancy between our results and Andersen & Broadie (2001)'s results that we must dwell on. They find that application of strategy 3 in the two factor deterministic volatility scenario will generate duality gaps all within 4 bps. We cannot support this finding for the 11 nc 1 Bermudan swaptions. It is only a minor discrepancy but for completeness we report our results in Table 6. The reductions in the duality gap from applying strategy 3 is not that dramatic. Notice that strategy 5 performs just as well as strategy 3 as we cannot reject that they are the same. Furthermore strategy 5 is much faster.

As the primal-dual simulation algorithm is computationally demanding it calls for the application of efficiency improvement. We have tested the effect of control variates on the estimate of the duality gap. These results are presented in Table 7. It appears that the duality gaps are reduced at a reasonable computational cost.

Table 6: Duality gaps in the two-factor scenario.

11 *nc* 1 *Strategy*	*Coupon* 0.08	0.10	0.12
1	18.9 (1.3)	20.0 (1.1)	14.1 (1.0)
2	13.5 (0.8)	13.5 (0.7)	9.7 (0.6)
3	8.4 (0.5)	6.8 (0.4)	3.4 (0.3)
4	13.1 (0.8)	14.0 (0.7)	7.8 (0.5)
5	8.8 (0.5)	6.5 (0.4)	3.9 (0.3)

Duality gaps for the 11 no-call 1 Bermudan swaption for different strategies and coupons in the two factor deterministic volatility scenario. The numbers have been computed using $n = 750$ and $m = 300$ in the primal-dual simulation algorithm.

Table 7: Effect of control variate on duality gaps.

	CV	*None*		*Cap*		*Reduc*	*Increase*
T_e	*Coupon*	$\hat{D}_0$	*Time*	$\hat{D}_0^{Cap}$	*Time*	$\hat{D}_0$	*Time*
3	0.08	0.19 (0.02)	00:33	0.09 (0.01)	00:45	53%	39%
3	0.10	0.24 (0.02)	00:44	0.11 (0.01)	00:53	54%	21%
3	0.12	0.30 (0.04)	00:48	0.22 (0.03)	00:54	27%	13%
6	0.08	2.15 (0.13)	03:58	0.88 (0.06)	05:15	59%	32%
6	0.10	1.99 (0.12)	06:04	0.89 (0.06)	06:54	55%	14%
6	0.12	0.99 (0.09)	07:09	0.43 (0.05)	07:41	57%	7%
11	0.08	14.17 (0.52)	23:15	7.63 (0.30)	28:03	46%	21%
11	0.10	11.96 (0.42)	34:50	6.71 (0.27)	38:40	44%	11%
11	0.12	6.38 (0.30)	41:50	3.96 (0.18)	44:36	38%	7%

The effect on the duality gaps when we use the cap as control variate and strategy 5. Numbers are from the two factor deterministic volatility scenario computed using $n = 1500$ and $m = 40$ antithetic paths in the primal-dual simulation algorithm. All Bermudan Swaptions have a lockout period of one 1 year.

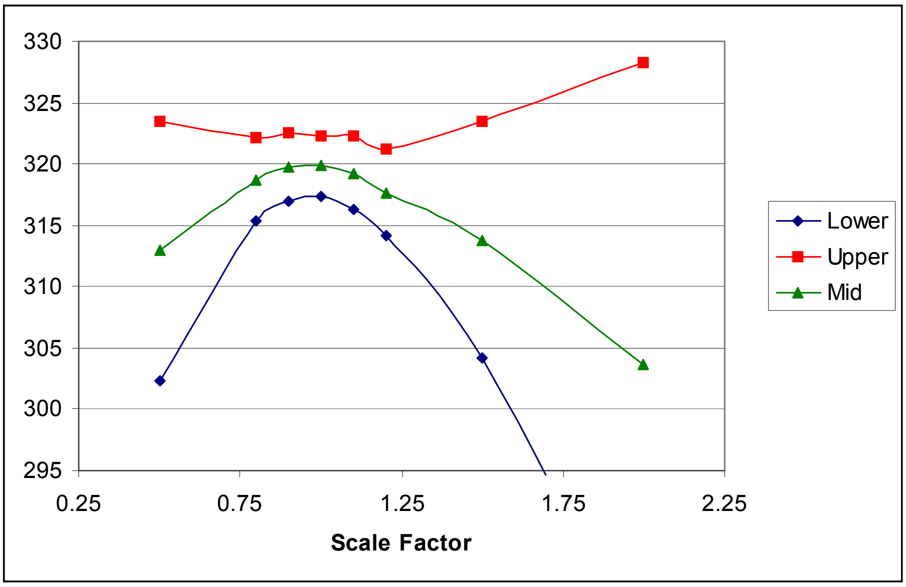

Figure 3: Illustration of the sensitivity to the excercise strategy. The figure shows the lower and upper bound for an ATM, 6-year Bermudan swaption with a 1- year lockout period as we scale the excercise barrier. The example is the deterministic volatility scenario using exercise strategy 1.

Notice that we have changed the relative amount of work used in the inner and outer computations relative to the previous calculations. This is due to the analysis in section 6.3. Particularly when combined with the control variate the estimates of the lower bound are not significantly different from the ones presented in e.g. table 6, but the standard deviations are lower and the calculation takes only about one third of the previous computations.

Tests have also been performed using several control variates in the inner loop, however remember that we determine the variance by minimizing β from OLS regression, so we have experienced problems due to the small number of inner loops.

6.7 Stochastic Volatility

In this section we investigate the effect of stochastic interest rate volatility on the size of the duality gap. The parameters in our test case have been taken from a scenario in Andersen & Brotherton-Ratcliffe (2001): $\theta_V = 1$, $\kappa_V = 1$, $\varepsilon_V = 1.4$ and $\psi(x) = x^{0.75}$.

We test whether the simple exercise strategy 1 still performs well when the forward curve dynamics exhibit stochastic volatility. This is done by examining the duality gap for various parameters in the variance of volatility process.

Notice that lower bounds are not comparable as the volatility of the forward rates varies as we vary the parameters. What could be a problem for the simple exercise strategy is that it does not distinguish whether the current volatility V_t is higher or lower than the long term mean. So a priori we would expect that the duality gap increases as the volatility of the variance process increases or when the mean reversion rate decreases. By estimating duality gaps we are able to asses the average present value loss that swaption holders incur from ignoring this feature of the yield curve dynamics.[6]

The results in Tables 8 and 9 confirm these a priori expectations. We have calculated the duality gap for a 6-year contract with 1-year lockout for various coupon rates of the underlying swap. Strategy 1 is reestimated using 50,000 (AS) paths in the presimulation for each value of the volatility parameter ε_V. Based on our observations in section 6.2.2, we apply the zero coupon bonds as a control variate to reduce the variance, as no closed form solutions for caps and swaptions are available[7], and we use $n = 1,500$ and $m = 300$ antithetic paths in the main simulation algorithm.

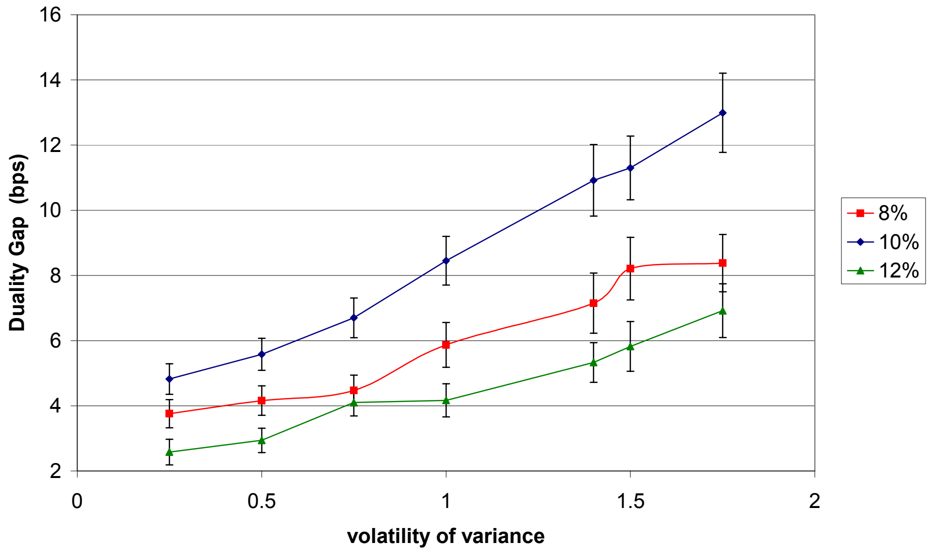

Figure 4:

As expected, higher volatility of the variance process increases the duality gap as the simple strategy fails to incorporate the volatility aspect in the exercise

[6] Longstaff et al. (2001*b*) argue that capturing the exact dynamics of the yield curve is very important for Bermduan swaptions.

[7] The approximations for European caps and swaptions given by Andersen & Brotherton-Ratcliffe (2001) are extremely precise and would likely be applicable as control variates as the bias would be very low relative to the variance. However, this remains to be investigated.

Table 8: Stochastic volatility - volatility of variance.

Coupon	ε_V	*Lower*	*StdLow*	$\hat{D}_0$	*Std* $\hat{D}_0$
0.08	0.25	750.03	1.50	3.76	0.22
0.08	0.50	751.24	1.58	4.16	0.23
0.08	0.75	748.41	1.56	4.47	0.24
0.08	1.00	749.87	1.73	5.87	0.35
0.08	1.40	747.76	1.73	7.15	0.47
0.08	1.50	747.11	1.74	8.21	0.49
0.08	1.75	755.15	2.01	8.38	0.45
0.10	0.25	317.53	2.98	4.82	0.24
0.10	0.50	313.46	3.02	5.58	0.25
0.10	0.75	311.03	3.17	6.70	0.31
0.10	1.00	311.30	3.33	8.45	0.38
0.10	1.40	314.03	3.42	10.92	0.56
0.10	1.50	309.74	3.58	11.30	0.50
0.10	1.75	326.87	3.99	12.99	0.62
0.12	0.25	124.10	2.43	2.58	0.20
0.12	0.50	132.74	2.77	2.94	0.19
0.12	0.75	123.18	2.63	4.10	0.21
0.12	1.00	124.22	2.89	4.17	0.26
0.12	1.40	122.64	2.90	5.33	0.31
0.12	1.50	131.23	3.19	5.82	0.39
0.12	1.75	143.53	3.42	6.92	0.42

This table illustrates the degree of inoptimality of the strategy 1. The contracts are 6 year contracts with 1 year lock out for three degrees of moneyness. As the volatility of the variance process ϵ_V increases the duality cap D_0 increases as well.
The basic scenario is the two-factor libor market model with a variance of the volatility process with the following initial value and parameters: $V(0) = \theta_V = 1$, $\kappa_V = 1$, $\psi(x) = x^{0.75}$. All zero coupon bonds were used as control variates for $n = 1500$ antithetic and $m = 300$. Barriers were estimated using $50,000$ (AS) paths.

decision. The same conclusion holds for the speed of mean reversion in the variance process. Higher mean reversion keeps the process closer to the long term mean.

Table 9: Stochastic volatility - mean reversion in variance.

Coupon	κ_V	*Lower*	*StdLow*	$\hat{D}_0$	$Std\hat{D}_0$
0.08	0.25	747.04	1.86	10.74	0.63
0.08	0.50	746.35	1.78	8.19	0.47
0.08	0.75	745.32	1.54	6.79	0.38
0.08	1.00	748.51	1.65	6.07	0.35
0.08	1.25	750.67	1.71	4.92	0.28
0.10	0.25	296.62	3.73	15.19	0.67
0.10	0.50	300.01	3.39	10.91	0.56
0.10	0.75	300.93	3.25	8.81	0.39
0.10	1.00	311.72	3.22	7.21	0.34
0.10	1.25	314.11	3.24	7.44	0.35
0.12	0.25	121.17	3.25	7.82	0.48
0.12	0.50	123.35	3.08	6.92	0.45
0.12	0.75	117.58	2.87	4.73	0.30
0.12	1.00	122.46	2.85	4.45	0.30
0.12	1.25	125.58	2.72	3.50	0.23

This table illustrates the degree of inoptimality of the strategy 1. The contracts are 6-year contracts with 1-year lock out for three degrees of moneyness. As the mean reversion of the variance process ϵ_V increases the duality cap D_0 decreases.
The basic scenario is the two-factor libor market model with a variance of the volatility process with the following initial value and parameters: $V(0) = \theta_V = 1$, $\epsilon_V = 1.4$, $\psi(x) = x^{0.75}$. All zero coupon bonds were used as control variates for $n = 1500$ antithetic and $m = 300$. Barriers were estimated using $50,000$ (AS) paths.

The sizes of the duality gaps are not very large even for rather extreme values of the parameters in the volatility process, but still economically significant. As already mentioned, Andersen & Brotherton-Ratcliffe (2001) provide very accurate approximations for European caps and swaptions that could be used to enhance strategy 1 to 5. Alternatively, one could use the Least-Square Monte Carlo approach of Longstaff & Schwartz (2001) including the current level of the variance process in the basis functions. As a final test we have implemented the LS-MC method following Bjerregaard Pedersen (1999). He concludes that a simple specification including a constant the first two powers of the intrinsic value and bankbook as well as the cross product in the regressions gave good results. We denote this strategy $LS1$. To account for stochastic volatility in the basis function we furthermore include the level of the variance process V_t in strategy $LS\ 2$. Table 10 illustrates that the Andersen approach performs slightly better than the $LS1$ approach. As expected the $LS\ 2$ strategy is superior to the other two as it picks up additional value using information in the variance level.

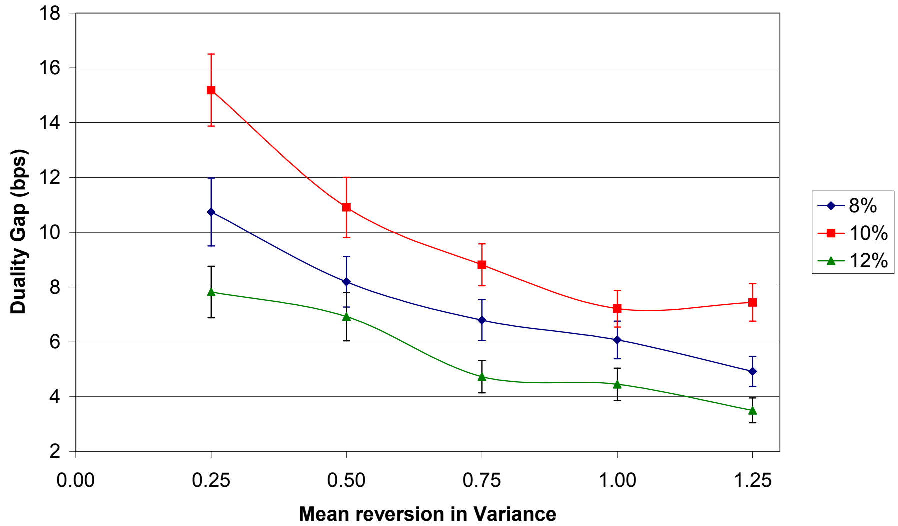

Figure 5:

7 Conclusion

This paper has addressed the issue of pricing Bermudan swaptions in a Libor market model. A control variate technique especially tailored for American options, which was recently proposed in Rasmussen (2002), was implemented and tested in the Libor market model. Furthermore, we demonstrated how to handle dividend paying control variates, and a range of controls were tested in the Bermudan swaption case. The results were reasonable showing reductions in standard deviations in the order of 3 to 5. The combination of the zero coupon bonds and caplets underlying the swaption performed well for a range of Bermudan swaptions.

A simplification of the strategies proposed in Andersen (2000) was demonstrated to give equally lower bounds at a significantly smaller computational effort. In particular, for the strategy resulting in the highest lower bounds the computation times were reduced to between 4% and 10% depending on moneyness of the option.

We also demonstrated the effect of the control variate technique on the duality gap from the Andersen & Broadie (2001) Primal-Dual algorithm, by applying the control variate technique in the nested simulations. The results showed significant improvements. However, we still note that the Primal-Dual algorithm is not suited for real time work. Still, it is extremely important due to the generality and reasonable computational effort. In particular it will enable us to determine when strategies are "good enough" so that we can use the lower

Table 10: Least-Square MC vs. Andersen approach under stochastic volatility.

Strategy			1		*LS* 1		*LS* 2	
T_s	T_e	*Coupon*	*Lower*	*Time*	*Lower*	*Time*	*Lower*	*Time*
1	6	0.08	750.4 (0.3)	0:14	746.7 (0.4)	0:16	751.2 (0.4)	0:16
1	6	0.10	313.2 (0.6)	0:27	311.3 (0.7)	0:29	315.7 (0.7)	0:29
1	6	0.12	130.8 (0.6)	0:33	128.5 (0.6)	0:34	132.1 (0.6)	0:34
1	11	0.08	1245.0 (0.8)	0:39	1237.9 (0.9)	0:48	1246.8 (0.9)	0:48
1	11	0.10	616.9 (1.2)	1:17	617.9 (1.4)	1:25	624.2 (1.4)	1:23
1	11	0.12	335.4 (1.1)	1:36	335.5 (1.2)	1:40	341.0 (1.2)	1:39

This table contains a simple comparison of the Andersen and the Least-Square approach. Strategy LS 1 incorporates the intrinsic value of the underlying swap and the bankbook. LS 2 furthermore incorporates the current level of the variance process V_t. Lower denotes the lower bound estimate. Time is calculation time in minutes and seconds. Numbers are based on 50,0000 AS paths with zero coupon bonds as controls in a two factor model with stochastic volatility. The strategies were estimated using 5,000 AS paths.

bound.

When setting up the Primal-Dual simulation algorithm one faces a variance-bias trade off. Our tests indicated that one should only increase the number of nested paths (reducing bias) with $\sqrt{2}$ when doubling the number of outer paths (reducing variance).

Finally, we considered an extended Libor market model with stochastic volatility developed in Andersen & Brotherton-Ratcliffe (2001), and demonstrate that this will increase the duality gap and make the enhanced strategies necessary also for short Bermudan swaptions. The Least-Square MC proposed in Longstaff & Schwartz (2001) was found to be a good substitute in case of stochastic volatility.

References

Andersen, L. (2000), 'A simple approach to the pricing of bermudan swaptions in the multifactor LIBOR market model', *Journal of Computational Finance* **3**(2), 5–32.

Andersen, L. & Andreasen, J. (1998), 'Volatility skews and extensions of the libor market model', *Working Paper, General Re Financial Products* .

Andersen, L. & Andreasen, J. (2000), 'Volatility skews and extensions of the libor market model', *Applied Mathematical Finance* **7**, 1–32.

Andersen, L. & Broadie, M. (2001), 'A primal-dual simulation algorithm for pricing multi-dimensional american options', *Working Paper: General Re Securities and Columbia University* .

Andersen, L. & Brotherton-Ratcliffe, R. (2001), 'Extended libor market models with stochastic volatility', *Working Paper: Gen Re Securities* .

Andersen, T. G. & Lund, J. (1997), 'Estimating continuous time stochastic volatility models of the short-term interest rate', *Journal of Econometrics* **77**, 343–377.

Bjerregaard Pedersen, M. (1999), 'Bermudan swaptions in the LIBOR market model', *Financial Research Department, SimCorp A/S* .

Boyle, P., Brodie, M. & Glasserman, P. (1997), 'Monte carlo methods for security pricing', *Journal of Economic Dynamics and Control* **21**, 1267–1321.

Brace, A., D. G. & Musiela, M. (1997), 'The market model of interest rate dynamics', *Mathematical Finance* **7**, 127–154.

Broadie, M. & Glasserman, P. (1997), 'Pricing american-style securities using simulation', *Journal of Economic Dynamics and Control* **21**, 1323–1352.

Clewlow, L. & Carverhill, A. (1994), 'On the simulation of contingent claims', *The Journal of Derivatives* **2**(2), 66–74.

Collin-Dufresne, P. & Goldstein, R. S. (2002), 'Do bonds span the fixed income markets? theory and evidence for unspanned stochastic volatility.', *Journal of Finance* **LVII**(4), 1685–1730.

Glasserman, P. & Kou, S. G. (1999), 'The term structure of simple forward rates with jump risk', *Working Paper, Columbia University* .

Glynn, P. W. (1994), 'Efficiency improvement techniques', *Annals of Operations Research* **53**, 175–197.

Glynn, P. W. & Whitt, W. (1992), 'The asymptotic efficiency of simulation estimators', *Operations Research* **40**(40), 505–520.

Haugh, M. B. & Kogan, L. (2001), 'Pricing american options: A duality approach', *Working Paper: Massachusetts Institute of Technology and University of Pennsylvania* .

Jamshidian, F. (1997), 'Libor and swap market models and measures', *Finance and Stochastics* **1**, 293–330.

Jamshidian, F. (1999), 'Libor market models with semimartingales', *Manuscript NetAnalytic Limited* .

Joshi, M. & Theis, J. (2002), 'Bounding bermudan swaptions in a swap rate market model', *QUARC Working paper* .

Longstaff, F. A., Santa-Clara, P. & Schwartz, E. S. (2001*a*), 'The relative valuation of caps and swaptions: Theory and empirical evidence', *Journal of Finance* **LVI**(6), 2067–2109.

Longstaff, F. A., Santa-Clara, P. & Schwartz, E. S. (2001*b*), 'Throwing away a billion dollars: The cost of suboptimal exercise strategies in the swaptions market', *Journal of Financial Economics* **62**(62), 39–66.

Longstaff, F. A. & Schwartz, E. S. (2001), 'Valuing american options by simulation: A simple least-squares approach', *The Review of Financial Studies* **14**(1), 113–147.

Miltersen, K., Sandmann, K. & Sondermann, D. (1997), 'Closed form solutions for term structure derivatives with log-normal interest rates', *Journal of Finance* **52**, 409–430.

Protter, P. (1995), *Stochastic Integration and Differential Equations*, Springer.

Rasmussen, N. S. (2002), 'Efficient control variates for monte-carlo valuation of american options', *Working Paper, The Aarhus School of Business* .

A Simulation

The simulation of the Libor market model is carried out under the measure $\mathbb{Q}$. We apply simple *log Euler Schemes* to (10) on a simulation time grid $0 \leq t_0 < t_1 < \cdots < t_{I+1} = T_{K+1}$. It is not necessary that the time grid is and the LIBOR maturity structure $T_0 < T_1 < \cdots < T_{K+1}$ are the same, but we will require that $\{T_0, T_1, \ldots, T_{K+1}\} \subseteq \{t_0, t_1, \ldots, t_{I+1}\}$ i.e. that the tenor dates are among the simulated dates. Thus we simulate the following

$$\begin{aligned}\hat{F}_k(t_{i+1}) &= \hat{F}_k(t_i)\exp\left(\lambda_k(t_i)\frac{\varphi\left(\hat{F}_k(t_i)\right)}{\hat{F}_k(t_i)}\sqrt{\hat{V}(t_i)}\left(\hat{\mu}_k(t_i)\Delta_i + \sqrt{\Delta_i}\varepsilon_i\right)\right) \\ &\quad \cdot \exp\left(-\frac{1}{2}\lambda_k(t_i)\left(\frac{\varphi\left(\hat{F}_k(t_i)\right)}{\hat{F}_k(t_i)}\right)^2 \hat{V}(t_i)\lambda_k(t_i)^T\Delta_i\right),\end{aligned}$$

where $t_{i+1} - t_i = \Delta_i$. For simplicity we will only use equally spaced tenor space and simulation grid where $\delta = \Delta$.So far our mapping function $n(t)$, used in the drift term (12), has been defined as left continuous.

So far this has not been a problem as in the continuous time setup this is without importance - for the obvious reasons this is not the case for the discrete time setup. Andersen & Andreasen (1998) argue that even though we operate with a left continuous mapping function in continuous time, we should use a right continuous mapping function in discrete time.

The V process in (11) is simulated using a Gaussian Ornstein-Uhlenbeck process proposed in Andersen & Brotherton-Ratcliffe (2001). We run n_i steps within each interval $[t_i, t_{i+1}]$ such that $t_j = t_i + j \cdot (t_{i+1} - t_i)/n_i$ $j = 1, .., n_i$

$$\hat{V}(t_{j+1}) = \theta_V + \left(\hat{V}(t_j) - \theta_V\right)e^{-\kappa_V(t_{j+1}-t_j)} + z_i\varepsilon_V\psi\left(V(t_i)\right)\sqrt{\tfrac{1}{2}\kappa_V\left(1 - e^{-2\kappa_V(t_{j+1}-t_j)}\right)}.$$

B Bias Variance Trade-off

Let $\hat{D}(n, m) = \sum_{i=1}^{n} X_i^m$ denote the point estimate for D using n inner simulations and m nested simulations. Assume that the estimator is consistent but biased

$$E\left(\hat{D}_m\right) = D_m \to D \; for \; m \to \infty.$$

We assume that the bias decreases with rate β as a function of m as

$$D_m - D \approx b \cdot \left(\frac{1}{m}\right)^\beta.$$

The expected time $T(m)$ per path using m inner simulations is assumed to be

$$T(m) = c \cdot m^\eta.$$

In our simulation experiment it is easy to realize that $\eta = 1$.

What we are looking for is an allocation rule. To determine n for a given m we postulate the following rule

$$n(m) = aT(m)^{\alpha} = ac^{\alpha} \cdot m^{\alpha\eta}$$

and use the estimator

$$\hat{D}(m) = \hat{D}(n(m), m).$$

The variance obtained using this rule

$$Var\left(\hat{D}(m)\right) = \frac{\sigma_m^2}{n(m)},$$

if we assume that $\sigma_m^2 \to \sigma^2$ for $m \to \infty$.

Given all these assumptions we let

$$\begin{aligned}
MSE\left(\hat{D}(m)\right) &= \left(\hat{D}(m) - D\right)^2 + Var(D_m) \\
&\approx \left(b \cdot \left(\frac{1}{m}\right)^{\beta}\right)^2 + \frac{\sigma_m^2}{n(m)} \\
&\approx b \cdot \left(\frac{1}{m}\right)^{2\beta} + \frac{\sigma_m^2}{aT(m)^{\alpha}} \\
&\approx b \cdot m^{-2\beta} + \frac{\sigma^2}{ac^{\alpha}} \cdot m^{-\alpha\eta}
\end{aligned}$$

So what we need to balance is

$$-2\beta = -\alpha\eta \Leftrightarrow \alpha = 2\frac{\beta}{\eta}.$$

This means that doubling the number of inner paths m we have

$$n(m) \propto m^{2\alpha\eta} = m^{2\beta}.$$

Hence, we have to estimate the rate β at which the bias decreases.

We propose the following procedure. First observe that

$$\hat{D}(n, m) - D = b \cdot m^{-\beta} + \frac{\sigma_m}{\sqrt{n}} z_n,\ z_n \sim N(0, 1).$$

As we don't know D, we provide an estimate of this based on relatively high values of n and m say n^* and m^*. Hence. the following equation gives a biased but consistent estimate of the bias using m inner paths given that n, n^* and m^* are high.

$$\begin{aligned}
\varepsilon_m &= \hat{D}(n, m) - \hat{D}(n^*, m^*) \\
&\approx b \cdot m^{-\beta} + \frac{\sigma_m}{\sqrt{n}} z_n - b \cdot (m^*)^{-\beta} + \frac{\sigma_m}{\sqrt{n^*}} z_n^* \\
&= b \cdot m^{-\beta} - \varepsilon_n^* \\
&\Leftrightarrow \\
\varepsilon_m + \varepsilon_n^* &= b \cdot m^{-\beta}
\end{aligned}$$

Assuming that the right side is positive (we can always increase $n, n^*$and m^* to ensure this) we take logarithms and apply a first order Taylor expansion to get

$$\ln(\varepsilon_m) + \frac{1}{\varepsilon_m}\varepsilon_n^* \approx \ln(b) - \beta \ln(m). \tag{16}$$

Now we can get a rough estimate of β by regression.

On the Suboptimality of Single-factor Exercise Strategies for Bermudan Swaptions

by

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On the suboptimality of single-factor exercise strategies for Bermudan swaptions

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Abstract

In this paper we examine the cost of using recalibrated single-factor models to determine the exercise strategy for Bermudan swaptions in a multi-factor world. We demonstrate that single-factor exercise strategies applied in a multi-factor world only give rise to economically insignificant losses. Furthermore, we find that the conditional model risk as defined in Longstaff, Santa-Clara & Schwartz (2001), is statistically insignificant given the number of observations. Additional tests using the Primal-Dual algorithm of Andersen & Broadie (2001) indicate that losses found in Longstaff et al. (2001) cannot as claimed be ascribed to the number of factors. Finally we find that for valuation of Bermudan swaptions with long exercise periods, the simple approach proposed in Andersen (2000) is outperformed by the Least Square Monte Carlo method of Longstaff & Schwartz (2001) and, surprisingly, also by the exercise strategies from the single-factor models.

JEL classification: C52; E43; E47; G12; G13;

Keywords: Bermudan swaption; American option; Least Square Monte Carlo; Libor Market Model; Model Risk; Model Calibration

1 Introduction

The notional amount of the contracts in the OTC market for interest rate derivatives contracts has been growing rapidly to an estimated $90 trillion. Of these

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interest rate swaps is by far the largest single group with outstanding contracts of \$68 trillion followed by interest rate options with \$12.5 trillion. The total market value is estimated to be \$2.5 trillion[1].

As discussed in e.g. Green & Figlewski (1999) the public is usually buying options leaving the dealer community with an overall short position. Several sources of risk have been identified in relation to derivatives trading, and in particular model risk[2] has been accentuated as the nominal amounts and complexity of derivative contracts have increased. As a consequence of this, the vast majority of the literature has been focusing on the choice of interest rate derivative models in relation to the hedging and valuation performance from the option writers view[3].

Another important aspect of model risk is faced by holders of American style options. By choosing a particular model, holders implicitly define their exercise strategy, but as stressed in Longstaff et al. (2001, p.43) *"...in an efficient market, an American option is only worth its market value to an investor who follows the optimal strategy"*. The expected present value of the cash flow from an American option is therefore less than the market value if anything but the optimal strategy is followed. The quantitative effect of using a slightly suboptimal exercise strategy is likely to differ from asset to asset, but it is important to stress that the quality of an exercise strategy can be measured in terms of the expected discounted cash flow obtained by following it. Better strategies lead to higher expected present values.

In this paper we concentrate on one of the most liquid American style interest rate derivatives, namely Bermudan swaptions. Recent studies by Longstaff et al. (2001) and Andersen & Andreasen (2001) have come to opposite conclusions about the significance of using slightly suboptimal exercise strategies in the market for Bermudan swaptions, and we therefore take these two papers as a starting point for a closer analysis.

In Longstaff et al. (2001) it is argued that the present value losses from using suboptimal exercise strategies are economically significant for swaption holders even if they are within bid/ask spreads, as they could have been avoided by using better strategies. In particular, they claim that the Wall Street practice of using the exercise strategies from single-factor models continuously recalibrated to market information, costs the holders of Bermudan swaptions billions of dollars as a whole. Furthermore, they argue that the present value cost conditional on making the wrong exercise decision is substantial and constitutes a new dimension to the potential effects of model risk. These conclusions are based on an extensive simulation experiment designed to replicate the Wall Street practice.

Contrary to Longstaff et al. (2001), the study by Andersen & Andreasen

[1] The estimates are based on the November 2002 press release from the Bank for International Settlements.

[2] See also Rebonato (n.d.*a*) for a good discussion of model risk.

[3] See e.g. Bühler, Uhrig-Homburg, Walter & Weber (1999), Green & Figlewski (1999), Hull & Suo (2001), Collin-Dufresne & Goldstein (2002), Driessen, Klaassen & Melenberg (2000), Gupta & Subrahmanyam (2002), Rebonato (n.d.*a*) etc.

(2001) argues that using exercise strategies derived from a best fit single-factor model only results in insignificant losses and provides a good mark-to-market of the Bermudan prices. Furthermore, they find that single-factor models, when calibrated appropriately to the prices of caps and European swaptions from a two-factor model, give Bermudan swaption prices that are slightly higher than those from the two-factor models.

Both these papers use simulation based valuation techniques in order to estimate the lower bounds for the true value of the Bermudan swaptions. However, they apply two different approaches for estimating the optimal exercise strategy. Longstaff et al. (2001) use the Least Square Monte Carlo technique by Longstaff & Schwartz (2001) (*LSM*), while Andersen & Andreasen (2001) apply the "simple approach" by Andersen (2000) (*LAM*).

In this paper we merge the approaches taken in these papers into a unified framework in order to facilitate a direct comparison. As the benchmark multi-factor model we set up a four-factor log-normal Libor market model based on a principal component analysis of the Euro forward-rate curve.

We start out by estimating the true Bermudan swaption values in this model using both the LAM and LSM exercise strategies. Interestingly, we find that the LAM strategy is outperformed by the LSM strategy, in particular when the exercise period is long. Furthermore, using the Primal-Dual algorithm by Andersen & Broadie (2001), we are able to conclude that the LSM strategies are in fact very close to being optimal.

Having established benchmark model prices for the Bermudan swaptions, we set up three different single-factor interest rate models including one spot-rate model (Black, Derman & Toy (1990)) and two forward-rate models within the Heath, Jarrow & Morton (1992) class (Ritchken & Sankarasubramanian (1995), Andreasen (2000)).

Using these we construct a total of six different single-factor exercise strategies and use these to compute lower bounds on a set of Bermudan swaptions. The application of efficiency improvement techniques allows us to give reasonably precise estimates. For three of these single-factor exercise strategies the lower bounds are virtually identical to the LSM values, leading us to conclude that losses from following exercise strategies from recalibrated single-factor models in the Bermudan swaptions market, are insignificant and economically irrelevant. This corresponds closely to the findings by Andersen & Andreasen (2001), but we stress that we have controlled for the optimality of the LSM strategy.

Turning to the conditional present value losses documented by Longstaff et al. (2001), we repeat the exact same simulation procedure but using our single-factor models. The best performing single-factor models perform reasonably well and the corresponding conditional losses relative to the LSM strategy are both negative and positive. More importantly, the losses are not significantly different from zero when using the same number of paths as in Longstaff et al. (2001). No standard errors on the loss estimates have been reported in Longstaff et al. (2001), but we find crude Monte Carlo estimation to be too inaccurate to conclude anything due to the sizes of the standard errors. The conditional losses are not examined by Andersen & Andreasen (2001).

We therefore test one single-factor exercise strategy in the Dual-Primal algorithm of Andersen & Broadie (2001), which is extremely slow due to the combination of nested simulations and computationally costly exercise decisions. In particular, we find that the upper bounds for the potential losses are very small and often lower than those from the full LSM approach.

This paper is organized as follows. First, we fix some notation regarding Bermudan swaptions. Secondly, we go through the simulation methodology and the benchmark multi-factor model. Then the exercise strategies are described in more detail and the ways to assess the true Bermudan prices are discussed. This is followed by a short outline of the single-factor models as well as their numerical implementation and calibration. All this lead us to the numerical results and finally we make our conclusions.

2 Notation and definitions

A *forward swap* is a standard financial contract where two parties agree to exchange a fixed coupon for a floating rate over a period of time. When paying/receiving fixed for floating this is termed a Payer/Receiver swap. Let t_s and t_e denote the forward starting date and the final maturity of the swap. Usually the first payment in the swap is fixed on t_s and paid on t_{s+1} and the last payment is fixed at time t_{e-1} and paid at time t_e. If we let $P(t,T)$ denote the time t value of a discount bond with maturity T, then the time t value of a payer swap with coupon θ is

$$S_{s,e}(t) = P(t,t_s) - P(t,t_e) - \theta \sum_{i=s}^{e-1} P(t,t_{i+1})(t_{i+1}-t_i),\ t \le t_s.$$

A standard payer *Bermudan* swaption $BS_{s,e}$ gives the holder the right, but not the obligation, to enter into the forward payer swap with a final maturity t_e and a coupon θ, on a set of times $t_s, ..., t_{e-1}$. As t exceeds tenor times we adjust the formula by only summing remaining payments. For this purpose we define $n(t)$ to be the mapping from a time into the swap's next reset time after t, i.e. $t_{n(t)} < t \le t_{n(t)}$

$$S_{s,e}(t) = P\left(t,t_{n(t)}\right) - P(t,t_e) - \theta \sum_{i=n(t)}^{e-1} P(t,t_{i+1})(t_{i+1}-t_i),\ t \le t_e.$$

That is, the intrinsic value X_t at an exercise time is $max\left(S_{n(t),e}(t),0\right)$. This intrinsic value is exactly the value of the European swaption $ES_{n(t),e}(t)$ of the swap. In this paper we define the Bermudan *premium* as the difference between the Bermudan swaption and the first to mature European swaption.

We make the standard assumptions of no arbitrage and complete and frictionless markets defined on a probability space $\left(\Omega, \{\mathcal{F}_t\}_{t\ge0}, \mathbb{P}\right)$, where $\{\mathcal{F}_t\}$ is a filtration and $\mathbb{P}$ the physical measure. Let $\mathbb{Q}$ denote the pricing measure induced by the numeraire asset β with the associated conditional expectation $E^{\mathbb{Q}}\left(\cdot|\mathcal{F}_t\right)$. The problem of pricing Bermudan swaptions is basically an optimal stopping problem,

$$BS_{s,e}\left(t\right)=\sup_{\tau\in\Gamma(t)}E^{\mathbb{Q}}\left(\frac{\beta_t}{\beta_\tau}X_\tau\middle|\mathcal{F}_t\right), \tag{1}$$

where $\Gamma\left(t\right)$ is the set of all $\mathcal{F}_t$-optional stopping times taking values in the set of exercise times from time t and on.

As should be well known (see e.g. Duffie (1996)), American style options should only be exercised when the intrinsic value X_t exceeds the continuation value of the option $BS^+_{s,e}\left(t\right)$, which defines the optimal exercise time

$$\tau^*=\inf_{t_s\leq t_i<t_e}\left(X_{t_i}\geq BS^+_{s,e}\left(t_i\right)\right). \tag{2}$$

Bermudan swaptions are in effect the right to choose between different swaps at different points in time. These core swap rates are the forward starting swap rates $SR_{i,e},\ i=s,..,e-1$ with same final maturity t_e as the swap in question. According to the literature (e.g. Andreasen (2000), Rebonato (2000)), it is not enough to know the right terminal distributions for the swap rates involved. An important part of pricing Bermudan swaptions is to have the right term-correlation (see appendix A.2) structure of the core swap rates.

3 Methodology

In this section we briefly summarize the simulation approach used by Longstaff et al. (2001) to investigate the effect of single-factor exercise strategies in a multi-factor world. They emphasize that even if various models have been calibrated to match a set of market prices from a multi-factor world perfectly, their implied exercise strategies will still be suboptimal.

The basic procedure goes like this. First, simulate a path from the multi-factor model to the first exercise date of the Bermudan swaption. Compute the yield curve and the European swaption prices from the benchmark model. Calibrate the single-factor model to this "market" information. Now, if exercise is implied by the single-factor model, we receive the discounted payoff from the benchmark model. If not, we advance the simulation to the next exercise date and so on until the Bermudan swaption has either been exercised or has expired. The expected present value loss is the difference between the average discounted payoff received by following a suboptimal exercise strategy and the discounted payoff received by following the optimal exercise strategy.

4 The benchmark multi-factor model

We choose as benchmark model a multi-factor log-normal Libor market model defined on a fixed tenor grid of 0.50 years. The Libor market models of Miltersen, Sandmann & Sondermann (1997), Brace & Musiela (1997) and Jamshidian (1997) have become increasingly popular among practitioners, in particular because they provide closed form solutions for both caps and European swaptions (though not in the same model). The reason for choosing a different benchmark model than Longstaff et al. (2001) is that the string model (see e.g. Santa-Clara & Sornette (2001)) applied in their analysis does not have closed form solutions for caps and European swaptions. In order to do the calibration of their one-factor models to the string model, they use the prices of ATM European swaptions extracted from an LSM regression. Although asymptotically unbiased, the size of the standard errors of these price estimates are also of importance but they are not reported in their study. In particular these price estimates are input to the calibration of the single-factor models and they could in principle distort the entire calibration. Therefore we use a Libor market model which does not involve such problems due to the existence of accurate approximations for the prices of European swaptions.

In the Libor market model the fundamental state variables are the discretely compounded forward-rates $F_k(t)$, $k = 0, .., K$ corresponding to a given tenor structure $t_0 < t_0 + \delta < t_0 + 2\delta < .. < t_K$. The dynamics of the forward rates under the spot Libor measure $\mathbb{Q}$ (see Jamshidian (1997)) can be written as

$$dF_k(t) = \mu(t, F(t))\,dt + F_k(t)\,\lambda_k(t) \cdot dW_t^{\mathbb{Q}},\ k = 1, .., K, \tag{3}$$

where $\mu(t, F(t))$ is a function of the entire forward rate curve and constructed to ensure no arbitrage. $W_t^{\mathbb{Q}}$ is an n-dimensional Wiener process and $\lambda_k(t)$ is a $n-$dimensional vector of the factor loadings of $F_k(t)$ on the Wiener process at time t. Notice that the dimension of the state variable is K, which could easily be as high as 120, when the last payment time is 30 year and there are quarterly payments.

4.1 Estimation of the benchmark model

We set up the benchmark model using the historical covariance matrix H of the percentage changes in the forward rates. This matrix can be written as

$$H = V^{\top} \Lambda V,$$

where Λ is a diagonal matrix of the eigenvalues (which are positive) and V the matrix of eigenvectors. We follow Longstaff et al. (2001) who make the identifying assumption that the implied covariance matrix Σ used in the swaptions market is

$$\Sigma = V^{\top} \Psi V,$$

where Ψ is a diagonal matrix of implied eigenvalues. This implicitly means that the factors that generate the historical covariance matrix also generate the implied covariance matrix, and if desired we could calibrate the model to match market quoted European swaptions on a given day using the diagonal in Ψ as free parameters. However, for the purpose of this study there is no need to match the implied swaptions volatilities on a particular date.

The historical covariance matrix used in this paper is estimated from daily changes in the forward Libor rates. The forward rates are calculated from yield curves estimated daily on a sample consisting of 3-month and 1-year EURIBOR and EUR swap rates using a cubic spline method[4]. The sample covers the period from Jan. 4th, 1999 to Oct. 3rd, 2002 which corresponds to 942 trade dates with observations on all 6 month forward Libor rates from 0.5 to 30 year. To reduce the data input we only include forward rates with maturities of 0.5, 1, 1.5,...., 5, 6, 7, 10, 15, 20, 25 and 30 year. It is well known that even if the covariance matrix has full rank, it is common that a relatively low number of factors are needed to describe the main part of the variance (e.g. Littermann & Scheinkman (1991)). Hence, if one chooses to use an $n-$factor model it is easily done by setting the remaining entries in the diagonal of Ψ equal to zero. Doing a principal component analysis on the covariance matrix estimated on the full sample, we find that the first factor explains 71% of the total variance, the second 13%, the third 7% and the fourth 5%. That is, the first four factors account for 96% of the total variation. The results from performing the same test on sub-samples consisting of the individual years, are similar to that of the full sample regarding both the explained fractions as well as the shape of the factors. We have included the factor loadings in the Appendix A.3. For forward rates not in the sample we interpolate linearly between the factor loadings.

For the test scenarios, the initial forward rates are set to 5 percent using discrete compounding. With these assumptions we are able to compute a matrix of implied swaption volatilities from the Benchmark model using the Andersen & Andreasen (2000) approximate swaption formula. As this matrix is input to the calibration procedure of the single-factor models, we present it in Table 1.

Table 1: Initial implied ATM swaption volatilites in the Benchmark model

Swaption	Swap Tenor						
Expiry	1	2	3	4	5	7	10
0.5	16.0	18.4	18.9	17.9	16.7	15.1	13.8
1.0	18.1	19.8	19.4	18.1	16.8	15.2	13.9
2.0	20.5	20.4	19.2	17.7	16.5	15.0	13.8
3.0	20.2	19.5	18.1	16.8	15.8	14.5	13.5
4.0	19.2	18.3	17.1	16.0	15.1	14.0	13.1
5.0	18.2	17.4	16.4	15.3	14.6	13.6	12.9
7.0	16.7	16.1	15.2	14.4	13.8	13.1	12.5
10.0	15.4	14.9	14.2	13.6	13.2	12.6	12.3

This table contains the initial implied volatilities for ATM European swaptions computed from the Benchmark model. Swaption Expiry denotes expiry of the swaption in years and Swap Tenor denotes the maturity of the underlying swap. Thus, a 2 into 5 year swaption is a 2 year option on a 5 year swap, so the final payment from the swap is in 7 years. The payment frequency of the swaps is semi-annual.

[4]Thanks to Peer Roer Pedersen, Jyske Bank for delivering the yield curve information.

4.2 Finding the true Bermudan values

In order to estimate the losses from following the exercise strategy from single-factor models in a multi-factor world, we need to have good estimates of the true value of the Bermudan swaption. We have implemented two of the most popular methods to find tight lower bounds for American options in a simulation model. These are the non-parametric "simple approach" proposed in Andersen (2000) (*LAM*) and the Least Squares Monte Carlo approach by Longstaff & Schwartz (2001) (*LSM).* Exercise strategies from these two approaches are used as input to a Dual-Primal simulation algorithm developed in Andersen & Broadie (2001) resulting in tight 95%-confidence intervals for the true value given that the strategies are close to being optimal.

Without going into detail about the Primal-Dual simulation algorithms by Andersen & Broadie (2001) (see also Haugh & Kogan (2001)), we here shortly discuss the relation to the present value losses. The main idea in these papers is to express the primal problem of (1) as a corresponding dual problem. Andersen & Broadie (2001) show that

$$BS_{s,e}(t) = \inf_{\pi} \left(\pi_t + E_t^{\mathbb{Q}} \left[\max_{s \in \mathcal{T}} \left(\frac{X_s}{\beta_s} - \pi_s \right) \right] \right),$$

where the infimum is taken over all $\{\mathcal{F}_t\}$ adapted $\mathbb{Q}$-martingales. They prove that the martingale component of the discounted Bermudan swaption price (which is supermartingale) is a solution, and they construct an approximation to this process for a given exercise strategy τ. In that way the duality gap Δ_t

$$\Delta_t = E_t^{\mathbb{Q}} \left[\max_{s \in \mathcal{T}} \left(\frac{X_s}{\beta_s} - \pi_s^{\tau} \right) \right]$$

works as a price measure of the suboptimality of the strategy τ. In fact, it measures the average worst case error along all paths given that we follow a suboptimal strategy τ. However, the analysis is slightly complicated by the fact that the estimate of Δ_t will be upward biased if the construction of π_s^{τ} requires nested simulations. Hence, we are only able to construct conservative estimates of the present value losses from following single-factor models using this method. Finally, it is important to stress that the duality gap is zero when the optimal strategy is used.

5 Exercise strategies

We start by listing the exercise strategies considered in this paper. First, we let a stopping time τ^i be defined as the first time an exercise indicator function $I^i(t)$ signals exercise,

$$\tau^i = \inf_{t \in \mathcal{T}} \left(I^i(t) = 1 \right),$$

where $\mathcal{T}$ is the set of possible exercise times, typically $s, ..., e-1$. The exercise indicator functions are in general allowed to be functions of the state variables.

Those considered here are all based on some form of approximation of the continuation value of the Bermudan swaption $BS^+_{j,e}(t_j)$ entering the definition of the optimal strategy in (2).

5.1 Barrier approach (LAM)

This is the preferred strategy from Andersen (2000), though with a minor modification. Andersen (2000) uses the maximal value of the still alive core European swaptions, but Jensen & Svenstrup (2002) show that it is much more computationally efficient to only use the first to expire, and there are no significant present value losses[5].

$$I^{LAM}(t_j) = \begin{cases} 1 & S_{j,e}(t_j)^+ > ES_{j+1,e}(t_j) + b(t_j) \\ 0 & else \end{cases} \tag{4}$$

Here the so-called barrier function $b(\cdot)$ is a real deterministic functions $R^+ \rightarrow R^+$. In words the strategy is to exercise the first time the intrinsic value exceeds the sum of the European swaption maturing at the next exercise time and a constant barrier. Notice, that the barrier b could be interpreted as the Bermudan premium.

5.2 Least Square Monte Carlo approach (LSM)

The LSM approach of Longstaff & Schwartz (2001) consists of approximating the continuation value by a linear function of conditioning variables, Y_t, computed from a d_t-vector Z_t of state variables

$$BS^+_{i,e}(t_i) \approx \alpha_{t_i} \cdot Y_{t_i},\ t_i \in \mathcal{T}.$$

Here $Y_t = g_t(Z_t)$, where $g_t(\cdot)$ is a vector function from $R^{d_t} \rightarrow R^{n_t}$ and α_t is an n_t-parameter vector. So the exercise indicator in the LSM case is

$$I^{LSM}(t_j) = \begin{cases} 1 & S_{j,e}(t_j)^+ > \alpha_{t_j} \cdot Y_{t_j} \\ 0 & else \end{cases}. \tag{5}$$

Following Longstaff et al. (2001), we use the values of the core swaps as the state vector Z_t, and for the transformation g_t we use the first three powers of the elements in Z_t and the cross products of the values between the current swap and forward swaps up to degree three. The parameter vectors α_t are estimated using OLS as proposed in Longstaff & Schwartz (2001)[6].

[5] We also tested the full strategy using the maximal value, and the results where virtually identical.

[6] In the numerical implementation we use singular value decomposition for the sake of computational stability (see e.g. Press, Flannery, Teukolsky & Vetterling (1989)).

5.3 Single-factor approach

Another way of estimating the continuation value is to use a more simple model, for example the single-factor models we present in the next section. For each of these single-factor models, now indicated with an asterisk, we define two exercise strategies by the following exercise indicators.

The most simple is based purely on the continuation value

$$I^{*V}(t_j) = \begin{cases} 1 & S_{j,e}(t_j)^+ > BS^*_{j+1,e}(t_j) \\ 0 & else \end{cases},$$

and simply signals exercise when the intrinsic value is larger than the continuation value in the simple model. This type of strategy we will denote with a subscript V for value based.

We also test a strategy where the next to mature core European swaption observed in the market works as a form of control variate. If, for some reason, the simple model does not match this European swaption, we suggest basing the exercise decision on the Bermudan premium estimate $BP^*_{j+1,e}(t_j) = BS^*_{j+1,e}(t_j) - ES^*_{j+1,e}(t_j)$ instead,

$$I^{*P}(t_j) = \begin{cases} 1 & S_{j,e}(t_j)^+ > ES_{j+1,e}(t_j) + BP^*_{j+1,e}(t_j) \\ 0 & else \end{cases}.$$

These strategies will be denoted with a subscript P for Premium based.

6 Single-factor interest rate models

The single-factor models we test in this study are the Black et al. (1990) short-rate model and two single-factor forward-rate models belonging to the Heath et al. (1992) class. In particular, we make sure that these are low-dimensional Markov models such that they can be implemented in a lattice. The BDT model is one of the single-factor models tested in Longstaff et al. (2001).

6.1 The Black, Derman & Toy model

The Black et al. (1990) (BDT) model is probably one of the most applied as well as one of the most severely criticized interest rate models. In the continuous time version of the model, the dynamics for the short rate r_t can be expressed as

$$d\ln r(t) = \left(\mu(t) + \frac{\sigma'_{BDT}(t)}{\sigma_{BDT}(t)}\ln r(t)\right)dt + \sigma_{BDT}(t)\,dZ_t.$$

We implement the model in a binomial tree using the forward induction algorithm described in Jamshidian (1991) to calibrate to the yield curve. One of the major drawbacks of this model is the lack of "real" mean reversion, and in order to match implied volatilities the volatility function will have to be decreasing $\sigma'_{BDT}(t) < 0$. This will effectively mean that the volatility disappears as time passes. We implement the BDT-tree with a volatility specification of $\sigma_{BDT}(t) = a + b\exp(-ct)$ and calibrate it to the initial prices of the European swaptions from the benchmark model using a brute-force search algorithm minimizing the relative squared price errors. The fit is not particularly satisfying with a root mean square error of 19.9%. However as we are interested in testing the degree of suboptimality, this further adds strength to the test. The parameter values found are $a = 13.8\%$, $b = 14.7\%$ and $c = 23.4\%$. Notice that these parameters will give rise to highly non-stationary short rate dynamics, as the volatility function is very steep.

6.2 A class of single-factor low dimensional Heath, Jarrow & Morton models

In this section we briefly go through a class of single-factor Heath, Jarrow & Morton models. We consider two particular parametrizations of the model, that will allow us to value Bermudan swaptions by solving a two-dimensional partial differential equation. The model derivation is included in order to explain the calibration procedure for the second version of the model. The two versions will be denoted RS and AN respectively.

Heath et al. (1992) show that under the risk neutral measure $\mathbb{Q}$ the dynamics of the continuously compounded forward rates $f_t(T)$ must satisfy the equation

$$f_t(T) = f_0(T) + \int_0^t \sigma(s,T) \int_s^T \sigma(s,u)\,du\,ds + \int_0^t \sigma(s,T)\,dW_s^{\mathbb{Q}}, \quad \forall t \leq T. \tag{6}$$

In this setting $W_s^{\mathbb{Q}}$ is a one dimensional wiener process under $\mathbb{Q}$ and $\sigma(t,T)$ is the instantaneous volatility function. In this paper we consider the class treated in e.g. Ritchken & Sankarasubramanian (1995), where the forward volatility is of the form

$$\sigma(t,T) = g(T)\,h_t. \tag{7}$$

$g(\cdot)$ is a deterministic function and h_t is some possibly stochastic process. Substituting this into (6) we get the following

$$f_t(T) = f_0(T) + \frac{g(T)}{g(t)}\left(x_t + y_t g(t)^{-1}\int_t^T g(s)\,ds\right), \tag{8}$$

where x_t and y_t are two state variables

$$\begin{aligned} x_t &= g(t)\int_0^t h_s^2 \int_s^t g(u)\,du ds + g(t)\int_0^t h_s dW_s^{\mathbb{Q}} \\ &= g(t)\int_0^t h_s^2 \int_s^t g(u)\,du ds + \int_0^t \sigma(s,t)\,dW_s^{\mathbb{Q}}, \\ y_t &= g(t)^2\int_0^t h_s^2\,ds = \int_0^t (h_s g(t))^2\,ds = \int_0^t \sigma(s,t)^2\,ds. \end{aligned}$$

By applying Leibnitz' rule the dynamics of the state variables can be seen to be

$$dx_t = \left(\frac{g'(t)}{g(t)} x_t + y_t \right) dt + g(t) h_t dW_t^{\mathbb{Q}},$$

and

$$dy_t = \left(2\frac{g'(t)}{g(t)} y_t + g(t)^2 h_t^2 \right) dt.$$

Let $\kappa(t) = -g'(t)/g(t)$ and let $\eta_t = g(t) h_t$ denote the volatility process of state variable x_t. If we let the η_t be a function of only the state variables x_t, y_t and time t, we have a two-dimensional Markov system. That is, let $\eta_t = g(t) h(t, x_t, y_y)$, then

$$\begin{aligned} dx_t &= (-\kappa x_t + y_t) dt + \eta_t dW_t^{\mathbb{Q}} \\ dy_t &= (\eta_t^2 - 2\kappa y_t) dt. \end{aligned}$$

Notice from equation (8) that the spot rate is $r_t = f_t(t) = f_0(t) + x_t$. From this it follows that the short rate evolves like

$$dr_t = \frac{\partial}{\partial T} f_0(t) dt + dx_t = \left(-\kappa x_t + y_t + \frac{\partial}{\partial T} f_0(t) \right) dt + \eta_t dW_t^{\mathbb{Q}}.$$

A closed form solution for zero coupon bonds in this model can be proved to be

$$\begin{aligned} P(t,T) &= \frac{P(0,T)}{P(0,t)} \exp\left(-x_t \int_t^T e^{-\int_t^u \kappa(u)du} ds - \frac{1}{2} y_t \left(\int_t^T e^{-\int_t^u \kappa(u)du} \right)^2 \right) \\ &= \frac{P(0,T)}{P(0,t)} \exp\left(-x_t G(t,T) - \frac{1}{2} y_t G(t,T)^2 \right). \end{aligned}$$

In our implementation we assume that the mean reversion function $\kappa(t)$ is constant, which implies that $g(t) = e^{-\kappa t}$ and $G(t,T) = \frac{1}{\kappa}\left[1 - e^{-\kappa(T-t)}\right]$.

Finally, under the usual conditions the Feynman-Kac theorem directly implies that the value function $V(t, x_t, y_t)$ of any interest derivate solves the following 2-dimensional partial differential equation (PDE)

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} V + \left[-r + (-\kappa x + y) \frac{\partial}{\partial x} + \tfrac{1}{2}\eta^2 \frac{\partial^2}{\partial x^2} + (\eta^2 - 2\kappa y) \frac{\partial}{\partial y} \right] V \qquad (9) \\ &= \frac{\partial}{\partial t} V + \left[-r + (-\kappa x + y) \frac{\partial}{\partial x} + \tfrac{1}{2}\eta^2 \frac{\partial^2}{\partial x^2} \right] V + \left[(\eta^2 - 2\kappa y) \frac{\partial}{\partial y} \right] V. \end{aligned}$$

This PDE is solved using a Craig & Sneyd split scheme described in Andreasen (2000), which is unconditionally stable and less prone to spurious oscillations, than the ordinary alternating direction implicit (ADI) scheme. Spurious oscillations are possible due to the lack of diffusion in the y-state variable.

6.2.1 Definition and Calibration - RS

First, we consider a simple, but time stationary, version within this class of models. We denote it *RS* as this version is often referred to as the Ritchken & Sankarasubramanian (1995) model. It is contructed by setting $\eta_t = \sigma r_t^\gamma$ such that

$$\sigma(t,T) = \sigma e^{-\kappa(T-t)}\ r_t^\gamma. \tag{10}$$

Unfortunately, we do not have closed form solutions for either caps or European swaptions in this version of the model, so we have chosen only to recalibrate the model initially, and as such using this model will only result in upper bounds on the losses. However, the benchmark model is also time stationary, so we expect that the volatility parameters extracted by calibrating to European swaptions are not varying much over time.

As we work with a log-normal Libor market model we let $\gamma = 1$. As shown in Heath et al. (1992) the forward rate volatility function should be bounded in order to be valid. However, as there are no closed form solutions for interest rate derivatives, and as the zero coupon prices are independent of the volatility function, we are free to switch to e.g a constant volatility at some cutoff level. To represent a simple time stationary model we make an initial brute force calibration of the two parameters on a sample of ATM European swaptions with maturities of 0.5, 1, 2, 3, 4, 5, 7 and 10 years written on swaps with maturities of 1, 2, 3, 4, 5, 7 and 10 years, a total of 56 swaptions. The objective in the optimization is equally weighted squared relative price errors. The optimal spot rate volatility found is $\sigma = 23.1\%$ and the mean reversion parameter $\kappa = 8.3\%$. Using only these two parameters we get a reasonable fit with a root mean square price error of 5.6%. The maximal percentage error is 21% which is due to the 0.5 into 1 year swaption that the model is unable to match. In general this model is incapable of producing a volatility hump like the one observed in the benchmark model[7].

6.2.2 Definition and Calibration - AN

This version of the model follows Andreasen (2000) and is accommodated to the pricing of Bermudan swaptions. Calibration is done individually for each of the Bermudan swaptions to the set of core European swaptions. It is a two-step procedure consisting of a separate calibration of the mean reversion level to fit the term correlation of the core swaptions, and after that a bootstrap of the variance structure of the underlying core European swaptions.

The volatility specification defining model *AN,* when pricing a T_e no-call T_s Bermudan swaption, is given by

$$\eta_t = \alpha(t)\, SR_{n(t),e}(t)^\gamma\,,\ t \leq t_{e-1} \tag{11}$$

[7] Rithcken & Chuang (1999) suggest another version that allows humped volatilities but in a three state Markov system. In our experience, this hump is not particularly important for the valuation of long swaptions.

where $\alpha(t)$ is a time dependent parameter vector and $SR_{n(t),e}$ the par swap rate corresponding to the next core swap. This means that the dynamics of the par swap rate under the k,e-swap measure $Q_{k,e}$ (see e.g. Jamshidian (1997)) is

$$dSR_{k,e}(t) = \frac{\partial SR_{k,e}}{\partial x}(t)\,\eta_t dW_t^{Q_{k,e}},\ t \leq t_k.$$

The first step in the Andreasen (2000) calibration procedure is to approximate the term-correlation structure of the $SR_{k,n}(t)$ and $SR_{l,p}(s)$ forward swap rates by

$$\begin{aligned} corr\left(SR_{k,n}(t), SR_{l,p}(s)\right) &\simeq corr(x(t), x(s)) \\ &\simeq \sqrt{\frac{\int_0^t \exp\left(2\int_0^u \kappa(v)dv\right)du}{\int_0^s \exp\left(2\int_0^u \kappa(v)dv\right)du}} \\ &= \sqrt{\frac{1-e^{-2\kappa t}}{1-e^{-2\kappa s}}},\ s > t \qquad (12) \end{aligned}$$

and calibrate this to the term-correlation structure of the benchmark Libor market model using a constant κ mean reversion function.

The second step utilizes the variance structure approximation

$$Var_{k,n}\left(SR_{k,n}(t_k)\right) \simeq \int_0^{t_k} \left(\frac{\partial SR_{k,n}}{\partial x}(u)\eta(u)\right)^2_{x=0,y=0} du,$$

which we can bootstrap to match the variance structure from the implied volatilities $\sigma_{j,e}^2$ of the core European swaptions for $j = s,..,e-1$

$$\sum_{i=s}^{j} \alpha_i^2 SR_{i,n}(0)^{2\alpha} \int_{t_i-1}^{t_i} \left(\frac{\partial SR_{j,e}}{\partial x}(u)\right)^2_{x=0,y=0} du = \sigma_{j,e}^2 SR_{j,e}(0)^2 t_j.$$

Here

$$\frac{\partial SR_{s,e}}{\partial x}(t) = SR_{s,e}(t)\left(\frac{-P(t,t_s)G(t,t_s)+P(t,t_e)G(t,t_e)}{P(t,t_s)-P(t,t_e)} + \frac{\sum_{j=s+1}^{e}\delta_j P(t,t_j)G(t,t_j)}{\sum_{j=s+1}^{e}\delta_j P(t,t_j)}\right).$$

Andreasen (2000) shows that when using this procedure the model matches the prices and skew of Bermudan swaption prices obtained in a Libor market using the LA approach.

We calibrate the term-correlation by running a one dimensional optimization over κ, where the objective is to minimize the squared absolute differences in the two term-correlation approximations. Figure 1 illustrates that we are able to match the term-correlations of the core swaptions. It should be noted that for very short swaptions the optimal mean reversion is sometimes negative, which of course is unacceptable. However, this usually happens when the term-correlation approximation in (12) is almost insensitive to the mean reversion κ, and we therefore choose to set the mean reversion to some minimal value of $\kappa = 0.5\%$.

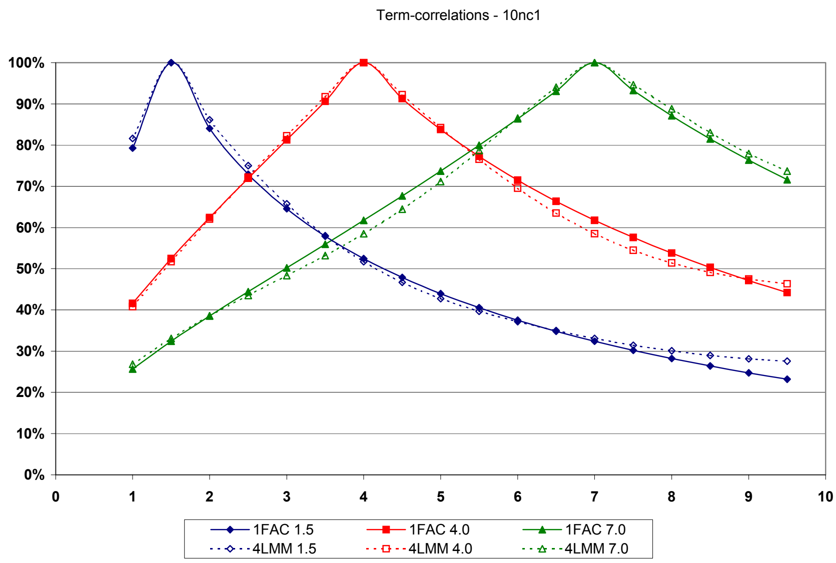


Figure 1: This figure shows an example of the calibrated term-correlations for a 10nc1 Bermudan swaption. On the first axis we have the forward times t_s of the core swap rates. The full lines are the approximations to the term-correlations between core swap rates starting at time 1.5, 4.0 and 7.0 in the benchmark model. The dotted lines are the fitted values in the single-factor AN model.

7 Numerical results

7.1 Simulation Setup

The computation of the expectation in equation (1) is done by Monte Carlo combined with various efficiency improvement techniques. The simulation is done by simulating a log-Euler discretized version of the forward rate dynamics in (3), see for example Andersen & Andreasen (2000).

Due to the high number of single-factor valuations required in order to make the exercise decisions, we are extremely interested in keeping the number of simulation paths as low as possible. To improve the efficiency we use both antithetic variables and control variates. While implementation of antithetic variables is straightforward, the control variate setup is more involved for Bermudan swaptions. We apply the Rasmussen (2002) technique for American options which consists of sampling the value of the controls at the exercise time. Jensen & Svenstrup (2002) demonstrate how to implement it with dividend paying assets. For Bermudan swaptions it was found that a combination of zero coupon bonds a cap control variates performed well across moneyness as well as maturities of both expiry and swap tenor. Both the LAM and the LSM approach require a presimulation of paths in order to determine the parameters in the exercise decision. Unless otherwise is stated we have used 25,000 antithetic paths (total of 50,000) for the presimulations. The pricing algorithms use another 250,000 antithetic paths (total of 500,000) to find the present values of the cash flow. Combined with the control variates the standard deviations of these estimates will be fairly small (around 0.1 bp).

7.2 Upper bounds for the Bermudan swaption values

In order to estimate the present value losses from following various exercise strategies, tight upper bounds for the real prices are required. Table 2 contains the lower bound estimates from both the LAM and the LMS exercise strategies as well as 95% confidence intervals for the true price. Notice that the upper bound is upward biased due to the effect of nested simulations in the dual-primal simulation algorithm. We should therefore not expect to be able to match them exactly. The duality gap $\hat{D}_0$ has been estimated using 1,500 antithetic paths in the outer loop and 300 antithetic paths in the inner loop, and to minimize the bias we have also used control variates (see e.g. Andersen & Broadie (2001)).

Table 2: Price estimates and 95% confidence intervals

			Low Bound		*Duality Gap*		*Upper CI* 95%	
t_s	t_e	θ	$\hat{L}_0^{LSM}$	$\hat{L}_0^{LAM}$	$\hat{\Delta}_0^{LSM}$	$\hat{\Delta}_0^{LAM}$	*LSM*	*LAM*
1	10	4%	767.4 (0.1)	762.7 (0.1)	0.5 (0.0)	5.9 (0.2)	768.1	769.0
1	10	5%	394.5 (0.1)	391.1 (0.1)	0.7 (0.1)	3.2 (0.2)	395.4	394.6
1	10	6%	205.9 (0.1)	204.2 (0.1)	0.2 (0.0)	1.7 (0.1)	206.4	206.2
3	10	4%	626.4 (0.1)	624.7 (0.1)	0.6 (0.0)	2.4 (0.1)	627.2	627.3
3	10	5%	355.6 (0.1)	354.2 (0.1)	0.5 (0.0)	1.9 (0.1)	356.3	356.3
3	10	6%	196.7 (0.1)	195.9 (0.1)	0.4 (0.0)	0.7 (0.1)	197.2	196.8
6	10	4%	359.7 (0.0)	359.8 (0.0)	0.3 (0.0)	0.2 (0.0)	360.0	360.1
6	10	5%	222.8 (0.0)	222.9 (0.0)	0.2 (0.0)	0.2 (0.0)	223.1	223.2
6	10	6%	135.1 (0.0)	135.1 (0.0)	0.2 (0.0)	0.2 (0.0)	135.4	135.3
1	15	4%	1080.1 (0.1)	1069.1 (0.1)	1.4 (0.1)	15.1 (0.5)	1081.7	1085.2
1	15	5%	578.2 (0.1)	570.8 (0.2)	1.4 (0.1)	8.6 (0.3)	579.9	580.1
1	15	6%	318.0 (0.2)	315.4 (0.2)	0.9 (0.1)	4.2 (0.2)	319.3	320.2

t_s, t_e and θ denote the lock out period, the final maturity and the coupon of the swaption. L_0^{LSM} and L_0^{LAM} denote the lower bound estimates from the Least Square Monte Carlo and the Andersen method, respectively. Both exercise strategies have been estimated using 25,000 AS paths in the presimulation and the price estimates are based on 250,0000 AS paths and control variates sampled using the Rasmussen (2002) method. The table also contains the estimated duality gaps as well as the upper end of the corresponding 95% confidence interval. All prices are in basis points.

The results in Table 2 demonstrate that the LAM exercise strategy generates slightly lower bounds than the LSM method. However, the duality gaps $\hat{\Delta}_0$ are still fairly small with a maximum of 15 basis points. The duality gaps for the

LSM method are very small indicating that this strategy is very close to being optimal. The table contains the upper limits in a conservative 95% confidence interval for the true prices based on the upward biased duality gaps.

It is particularly interesting that the LAM method fails to pick up the last basis points for the Bermudan swaptions with long exercise periods. These findings illustrate the importance of the primal-dual algorithm of Andersen & Broadie (2001) as it will allow us to detect exercise strategies that are far from being optimal.

Finally, these results could in principle invalidate some of the conclusions made in Andersen & Andreasen (2001) regarding the lower prices in their two-factor model than in their single-factor model (on page 26 they report the numbers one to five basis points). To test that these differences could not be ascribed to the suboptimality of their two-factor exercise strategy, we have reconstructed their Table 6 but using the LSM exercise strategy[8]. The differences now decrease to about one to three basis points. It is hard to say whether differences of this size support their claim that prices are decreasing in the number of yield curve factors and not just due to different model dynamics. On the other hand it only adds strength to their claim that a single-factor model can be brought to replicate the values from a two-factor model.

7.3 Swaption values in the single-factor models

To provide some intuition about the fit of the single-factor models in this paper, we present Table 3, which shows the "market" prices from the Libor market model for a range of European swaptions. It also includes the prices of the Bermudan and European swaptions computed at time zero using the single-factor models fitted to the Libor market model as described above. Finally, the Bermudan exercise premiums are included.

The European swaption prices indicate how well the single factor models fit the Libor market model. In fact, as already noted, the BDT model's fit is quite poor, in particular for OTM swaptions. The fit of the time stationary RS model is reasonable, but not perfect either. The AN model has a very nice fit to European swaption prices due to the calibration and especially when the lock out period is short. However, we see that the fit for the European swaptions deteriorate slightly as the lock out period t_s increases. This is probably due to the approximations in the bootstrap calibration, where we keep the state variables fixed, and it suggests that for long European swaptions a better approximation should be used for a really tight mark-to-market[9]. However, for the application in this paper this should not be an issue, as we continuously recalibrate this version once the lock out period is over. Of course this is also a concern for the remaining core swaptions, but again a slack in the calibration is just another drawback for the single-factor model.

[8] Table is not included - but is available from the author upon request.

[9] For example, one could try computing the approximation keeping only $x = 0$, as y is locally deterministic it could be approximated using the forward curve.

Table 3: Initial swaption prices from the single-factor models.

			LM	*BDT*			*RS*			*AN*		
t_s	t_e	θ	EU	BS	EU	PRE	BS	EU	PRE	BS	EU	PRE
1	10	4%	694.0	808.6	715.4	93.2	780.7	698.3	82.3	763.8	693.7	70.2
1	10	5%	193.9	436.7	255.3	181.4	418.5	218.4	200.1	369.6	191.9	177.7
1	10	6%	25.3	234.3	64.0	170.3	232.1	44.7	187.4	173.5	25.6	147.9
3	10	4%	561.4	644.5	593.0	51.5	630.3	566.9	63.4	615.2	557.2	58.0
3	10	5%	251.2	377.0	301.7	75.3	370.7	273.4	97.3	332.6	246.0	86.7
3	10	6%	93.8	214.6	138.2	76.4	218.0	121.8	96.2	170.4	90.7	79.7
6	10	4%	337.4	361.6	348.6	13.0	359.1	337.6	21.5	347.0	327.7	19.4
6	10	5%	192.2	224.8	207.6	17.2	228.3	200.6	27.7	204.7	179.3	25.4
6	10	6%	103.8	136.3	118.6	17.6	145.2	117.9	27.3	116.5	91.9	24.6
1	15	4%	960.4	1141.4	981.8	159.6	1092.1	960.7	131.5	1068.7	960.2	108.5
1	15	5%	248.7	646.6	325.6	321.1	602.6	263.2	339.5	544.6	246.9	297.7
1	15	6%	25.7	366.1	70.4	295.6	349.7	41.7	308.1	281.9	26.4	255.5

This table contains the prices in basis points of a set of Bermudan payer swaptions from the single factor models considered. t_s, t_e and θ denote the lock out period, the final swap payment and swap coupon. LM denotes the benchmark 4-factor Libor Market model, BDT the Black, Derman and Toy, *RS* the single-factor Rithcken & Sankarasubramanian model and *AN* the Andreasen model. BS and EU denote the value of the Bermudan and European swaption respectively, while PRE is the Bermudan exercise premium.

Figure 2: Initial percentage errors in the Bermudan values from the single-factor models

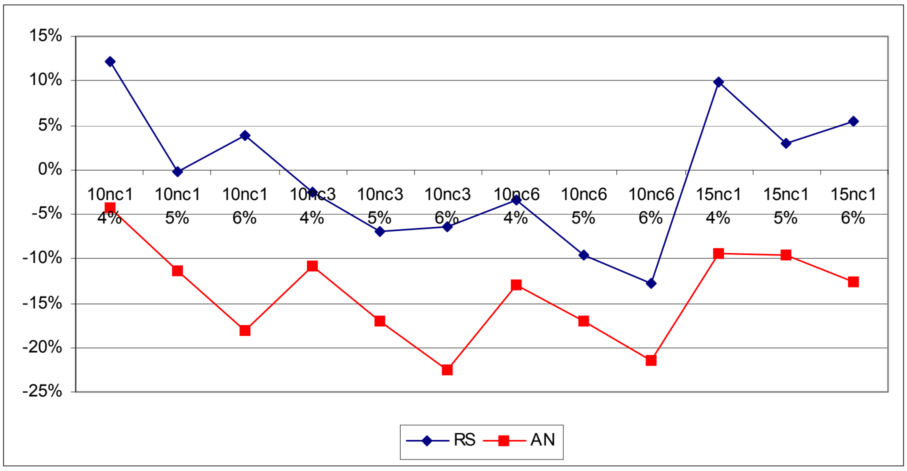


Illustration of the initial percentage errors of the single-factor forward-rate models RS and AN. Interestingly the model with the best fit to the European swaptions has larger price errors for the Bermudan swaptions.

7.4 Losses from following single factor exercise strategies

Having described the benchmark model, the three-single factor models, as well as their calibration and exercise strategies, we present the results from the full simulation algorithm. Due to the computationally demanding procedure we keep the number of paths relatively low. We use 4,000 antithetic paths combined with the control variates already described.

The results presented in Table 4 confirm the findings in Andersen & Andreasen (2001). As the 95% confidence intervals overlap with the LSM estimates it is hard to be very precise about the exact sizes of the losses (or gains) from following single-factor exercise strategies. But the bounds we are able to put on the losses are extremely small for most of the single-factor models. As expected, the Black et al. (1990) model has the worst performance, and overall the prices are lower than the LSM values. The performance of the single-factor forward-rate models, on the other hand, is comparable to that of the LSM approach. All single factor models perform better than the LAM approach. Overall, standard errors are at most of the order of a single basis point - but nothing indicates that the single-factor model's values are systematically below the LSM prices. Notice that these standard errors error actually are quite small due to the control variates. To obtain a similar precision with outcontrol variates we would roughly need 50,000 antithetic paths (100,000 total). So, even with 50,000 antithetic paths we cannot reject that the losses from following single-factor exercise strategies are zero for the best performing single-factor models.

Even if the losses cannot be distinguished from zero it is interesting to study their relative performance a little closer, as there appears to be systematic differences. In figure 3 we have computed the loss measured relative to the Bermudan premium. Notice that these estimates are much more affected by the sampling error and are to be considered with caution, but still they indicate that there are systematic differences in the relative performance of the models. In the paper of Longstaff et al. (2001) a set of common paths is used to compute these numbers. If we make the same comparison measuring the losses relative to the total contract value, instead of just the Bermudan premium, the maximal loss is only 0.03%.

Figure 3 illustrates that on an overall basis the exercise strategies using the Bermudan premium, combined with the observed value of the first to expire core European swaption, is performing slightly better than the value based exercise strategies. In some way this approach works as a simple form of calibration to the first to mature European swaption.

7.5 Conditional present value costs

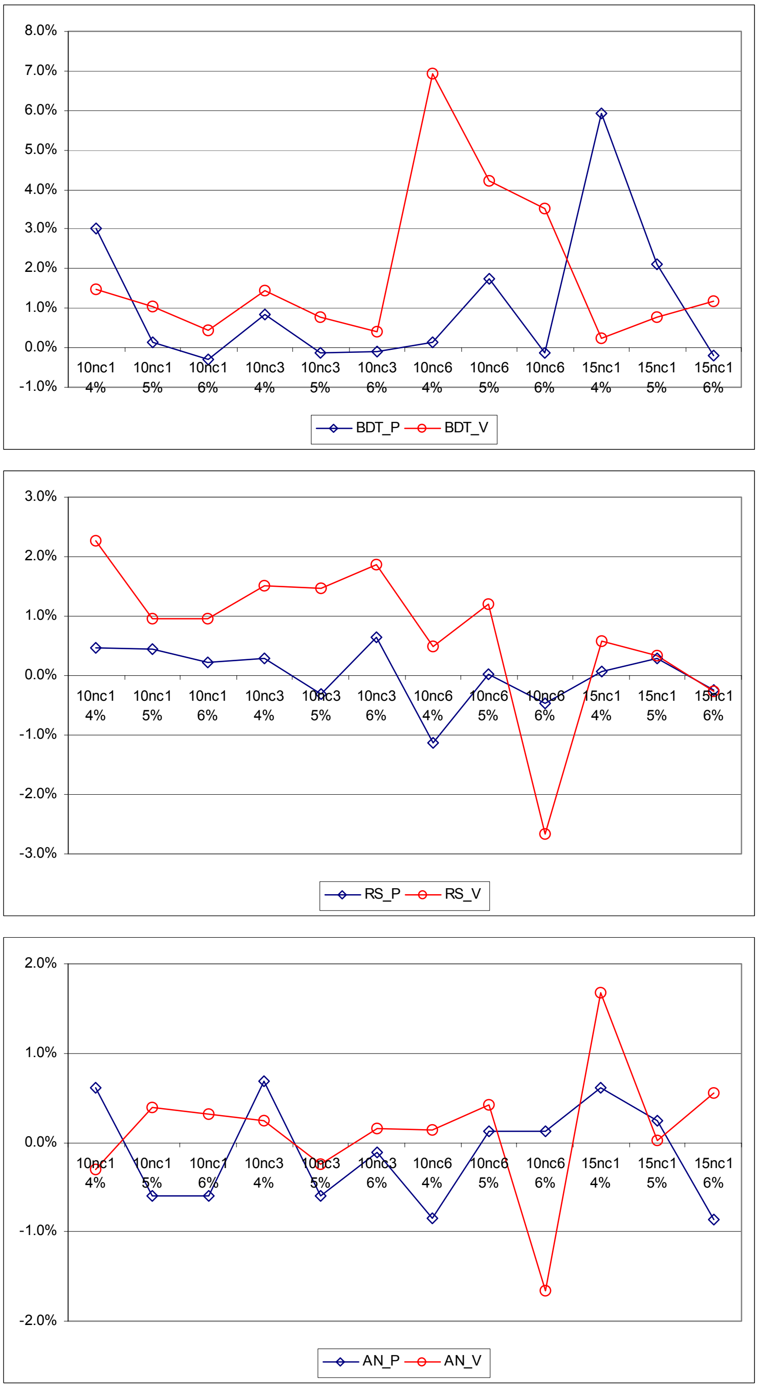
In this section we present the results in terms of the conditional present value costs as defined in Longstaff et al. (2001). The basic idea is to compare the expected losses on a common set of paths, where the single-factor strategies differ from the LSM strategy. Their argument is that we should measure the strategies where the exercise decision is a tough call, and focus more on the

Table 4: Comparison of single-factor exercise strategies

t_s	t_e	θ	LSM	$Single-factor\ exercise\ strategies$					
				BDT_V	BDT_P	RS_V	RS_P	AN_V	AN_P
1	10	4%	767.4	766.3	765.1	765.7	767.0	767.6	766.9
			(0.1)	(0.5)	(0.6)	(0.6)	(0.5)	(0.5)	(0.5)
1	10	5%	394.5	392.4	394.2	392.6	393.6	393.7	395.7
			(0.1)	(0.7)	(0.7)	(0.8)	(0.7)	(0.7)	(0.7)
1	10	6%	205.9	205.2	206.4	204.2	205.5	205.4	207.0
			(0.1)	(0.8)	(0.8)	(0.8)	(0.8)	(0.8)	(0.8)
3	10	4%	626.4	625.5	625.9	625.4	626.2	626.3	626.0
			(0.1)	(0.4)	(0.4)	(0.4)	(0.4)	(0.4)	(0.4)
3	10	5%	355.6	354.9	355.8	354.1	356.0	355.9	356.3
			(0.1)	(0.5)	(0.6)	(0.6)	(0.6)	(0.5)	(0.5)
3	10	6%	196.7	196.3	196.8	194.7	196.0	196.5	196.8
			(0.1)	(0.6)	(0.7)	(0.7)	(0.7)	(0.7)	(0.7)
6	10	4%	359.7	358.1	359.6	359.6	359.9	359.6	359.9
			(0.0)	(0.2)	(0.2)	(0.2)	(0.2)	(0.2)	(0.2)
6	10	5%	222.8	221.5	222.3	222.5	222.8	222.7	222.8
			(0.0)	(0.2)	(0.2)	(0.2)	(0.2)	(0.2)	(0.2)
6	10	6%	135.1	134.0	135.2	136.0	135.3	135.6	135.1
			(0.0)	(0.3)	(0.3)	(0.3)	(0.3)	(0.3)	(0.3)
1	15	4%	1080.1	1079.8	1073.0	1079.4	1080.0	1078.0	1079.3
			(0.1)	(0.8)	(0.9)	(0.9)	(0.8)	(0.8)	(0.8)
1	15	5%	578.2	575.6	571.3	577.1	577.3	578.1	577.4
			(0.1)	(1.2)	(1.2)	(1.2)	(1.2)	(1.2)	(1.2)
1	15	6%	318.0	314.6	318.6	318.8	318.7	316.4	320.6
			(0.2)	(1.2)	(1.2)	(1.2)	(1.2)	(1.2)	(1.2)

This table contains the results from the simulation approach. For a set of Bermudan swaptions we report the present value of the cash flows received by following the optimal strategy and exercise strategies from the described single-factor models. t_s, t_e and θ denote the lock out time, the time of the swap maturity and coupon. LSM denotes the Least square strategy, which we have shown to be close to optimal. BDT, RS and AN denote the single-factor exercise strategies and subscripts V and P denote whether they are based on the Bermudan Value or Bermudan Premium plus observed European swaption. Numbers in parentheses are standard deviations. All present values are in basis points. The LSM values are based on 25,000 AS paths in the presimulation and 250,000 AS paths combined with control variates. The single-factor prices are based on 4,000 AS paths and control variates.

Figure 3: Losses relative to the Bermudan premium



This figure plots estimates of the losses for the various single-factor exercise strategies. Percentage errors have been computed relative to the Bermudan exercise premium.

net present values on the paths, where we make "wrong" exercise decisions. So far our results have supported the findings in Andersen & Andreasen (2001). However, they did not investigate conditional present value costs.

Following Longstaff et al. (2001) we consider the conditional present value loss when the single-factor models signal exercise later/earlier than LSM,

$$PVL^i_{Late} = E_0^{\mathbb{Q}}\left(\frac{X_{\tau^{LSM}}}{\beta_{\tau^{LSM}}} - \frac{X_{\tau^i}}{\beta_{\tau^i}}\left|\tau^i > \tau^{LSM}\right.\right),$$

and

$$PVL^i_{Early} = E_0^{\mathbb{Q}}\left(\frac{X_{\tau^{LSM}}}{\beta_{\tau^{LSM}}} - \frac{X_{\tau^i}}{\beta_{\tau^i}}\left|\tau^i < \tau^{LSM}\right.\right).$$

When we define these expectations we implicitly assume that we are not conditioning on null sets. We could easily define a stopping time τ for which one of these is undefined. Furthermore, computing these by simulation can also give rise to problems, especially when the strategies in question are very similar, as the number of observations will be low. This effect is particularly unfortunate as we are bound to keep the number of paths low. 5,000 paths were used to estimate the conditional present value losses, and the results are presented in Tables 5 (similar tables for the other models have been included in Appendix A.4). Notice that these tables include the standard deviations of the loss estimates which are not reported by Longstaff et al. (2001). Furthermore, all losses reported by Longstaff et al. (2001) are negative, which we would interpret as gains.

In Table 5 we have reported the summary statistics for the exercise strategies AN_V and LSM in the multi-factor Libor market model. Columns four and five show that the number of exercises using the single-factor model is very close to the number using the LSM from the multi-factor model. The percentage of paths where the two models signal that it is optimal to exercise at the same time, is ranging from 88.3% to 94.6%, which is not particularly impressing. Despite this, present value losses associated with these differences, presented in the last columns, are overall not different from zero on a 95% level of significance. Furthermore, the signs of the losses are negative as well as positive across different swaptions. We therefore conclude that we cannot reject that the single-factor model performs just as well as the LSM strategy based on these estimates. Similar tables for the other single-factor models have been included in Appendix A.4. Not all of these perform as well as the AN models and particularly the BDT models perfom worse than the others, which is somehow what we expected given the poor dynamics and the poor initial fit to the European swaptions. On an overall basis the standard errors of the conditional loss estimates are very large.

In our opinion these results merely illustrate that this way of estimating the conditional present value loss is not very precise. The number of paths required in order to obtain an accuracy that would enable us to say somthing meaningful about present value losses using this approach, would require practically months of CPU time (at least in our implementation).

Table 5: Comparison of the single-factor AN and four-factor Libor market model exercise strategies

Swaption			Probability of Exercise %		Single-factor Exercises						
					Same			Early		Late	
t_s	t_e	θ	LSM	AN_V	Early	time	Late	Loss	Std	Loss	Std
1	10	4%	89.9	89.5	4.3	88.3	7.4	-19	(14)	30	(18)
1	10	5%	68.6	68.0	5.3	90.2	4.5	5	(11)	3	(21)
1	10	6%	46.1	45.6	3.9	93.8	2.4	17	(11)	24	(23)
3	10	4%	84.7	84.5	6.6	91.9	1.6	-1	(8)	-33	(22)
3	10	5%	63.9	63.8	5.8	91.8	2.4	-2	(10)	13	(18)
3	10	6%	46.7	46.1	4.0	93.1	2.9	-15	(11)	7	(20)
6	10	4%	78.0	78.1	9.5	89.0	1.6	-10	(6)	20	(10)
6	10	5%	58.4	58.0	5.5	92.9	1.7	-11	(6)	5	(9)
6	10	6%	41.5	41.3	4.2	94.6	1.2	-4	(8)	-17	(11)
1	15	4%	89.8	89.9	5.6	90.0	4.3	-19	(14)	4	(26)
1	15	5%	70.5	70.0	6.9	88.4	4.7	-23	(13)	12	(24)
1	15	6%	53.0	52.2	4.0	91.1	4.9	-2	(12)	10	(24)

This table reports summary statistics for the single-factor exercise strategy and the Least Square Monte Carlo exercise strategy in the multi-factor Libor market model. Probability of Exercise represents the total percentage of paths for which the swaption is exercised. Also the percentage of paths where the single-factor models signals exercise earlier, at the same time, or later than the multi-factor model are included. The present value losses of exercising when the single-factor model implies it is optimal at an earlier time than the four-factor model is the difference between the immediate value of exercise and the present value of cash flows generated by following the multi-factor strategy, averaged over all paths where the single-factor implies exercise earlier than. Similarly, for the present value loss of exercising when the single-factor model implies that exercise is optimal at a later time than the four-factor model. All costs are expressed in basis points. Values are based on 5,000 simulated paths of the term structure.

We have considered the application of various efficiency improvement techniques (see e.g. Boyle, Brodie & Glasserman (1997)), but it is not straight forward to come up with a good solution. Antithetic sampling only works well when the sampled variables are negatively correlated given two "mirror" paths. But in this case there is no reason to believe that any differences in the conditional losses for two mirror paths are highly negatively correlated. Furthermore, in order to apply the control variate technique, we are required to find some stochastic variable that is highly correlated with the loss conditional on the exercise strategies differing, and with a first moment we can compute. Such a stochastic variable is not likely to exist.

7.6 Assessing the suboptimality using the primal-dual algorithm

In order to provide more precise estimates of the conditional present value losses we compute the duality gap for the RS_P single-factor model using the dual-primal algorithm. Testing the single-factor exercise strategies using this algorithm is a huge computational task, due to the combination of nested simulations and an exercise strategy that requires us to solve a 2-dimensional PDE. Therefore, we are forced to keep the simulation paths for the dual-primal algorithm low. Computation time is also the reason as to why we do not test the AN exercise strategies even though these performed better in the previous tests.

Before commenting on the results we stress again that these estimates are in fact upward biased due to the relative low number of paths, which is also seen by comparing with the results in table 2.

The results in Table 6 demonstrate that given the same number of paths in the primal-dual simulation algorithm, the RS_P single-factor exercise strategy generally outperforms the multi-factor LSM approach.

Even if we use the upper end of a conservative 95% confidence interval for the losses and measure relative to the Bermudan exercise premium (which is much smaller than the total Bermudan value) we are still far from the percentage losses reported in Longstaff et al. (2001). The duality gaps have also been illustrated in Figure 4.

8 Discussion

Longstaff et al. (2001) stress the importance of studying single- versus multi-factor models in the correct way, namely by comparing the received cash flows in the true model by using only the exercise decisions from single-factor models. The ability of a single-factor model to match the caps and European swaptions from a multi-factor model by calibration does not necessarily mean that the exercise decision for Bermudan swaptions computed within such a model is optimal.

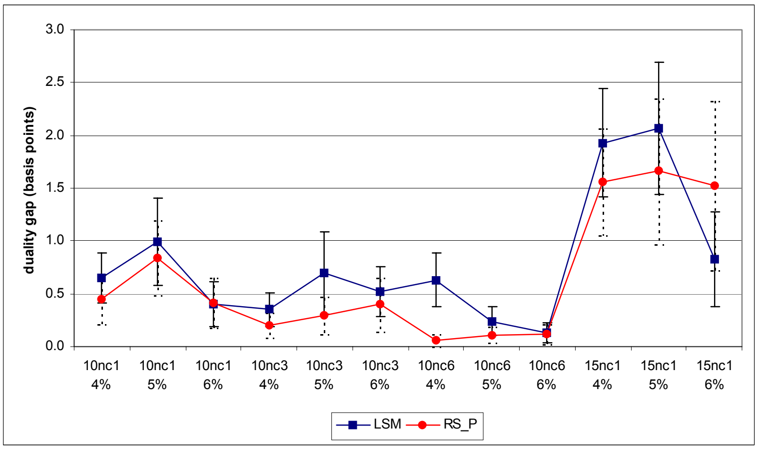
As our results show, even for the worst performing single-factor exercise strategy, the expected losses are very limited.

Table 6: Comparison of the duality gap for the multi-factor LSM and the RS_P single-factor exercise strategy

t_s	t_e	θ	Bermudan Exercise Premium	Duality Gap D_0^{LSM}	Duality Gap $D_0^{RS_P}$	Pct. Loss LSM	Pct. Loss RS_P	Conservative Pct. Loss LSM	Conservative Pct. Loss RS_P
1	10	4%	73	0.7 (0.1)	0.5 (0.1)	0.9	0.6	1.2	0.9
1	10	5%	201	1.0 (0.2)	0.8 (0.2)	0.5	0.4	0.7	0.6
1	10	6%	181	0.4 (0.1)	0.4 (0.1)	0.2	0.2	0.3	0.4
3	10	4%	65	0.4 (0.1)	0.2 (0.1)	0.5	0.3	0.8	0.5
3	10	5%	104	0.7 (0.2)	0.3 (0.1)	0.7	0.3	1.0	0.5
3	10	6%	103	0.5 (0.1)	0.4 (0.1)	0.5	0.4	0.7	0.6
6	10	4%	22	0.6 (0.1)	0.1 (0.0)	2.8	0.3	4.0	0.5
6	10	5%	31	0.2 (0.1)	0.1 (0.0)	0.8	0.4	1.2	0.6
6	10	6%	31	0.1 (0.1)	0.1 (0.1)	0.4	0.4	0.7	0.7
1	15	4%	120	1.9 (0.3)	1.6 (0.3)	1.6	1.3	2.0	1.7
1	15	5%	329	2.0 (0.3)	1.7 (0.4)	0.6	0.5	0.8	0.7
1	15	6%	292	0.8 (0.2)	1.5 (0.4)	0.3	0.5	0.4	0.8

This table contains the duality gaps from the multi-factor Least square Monte Carlo method (LSM) and the single-factor *RS* exercise strategy. The Duality Gap is a measure of the expected losses from following a given strategy. The Bermudan Exercise Premium denotes the difference between the Bermudan- and the European swaption values in the multi-factor model using the LSM strategy. The Pct. Loss is the duality gap in percent of the Bermudan exercise premium. Conservative Pct. Loss denotes the duality gap plus 1.96 times the standard deviation of the duality gap relative to the Bermudan exercise premia. Duality gaps have been estimated using 200 AS paths in the outer simulation and 25 AS paths in the inner. Furthermore, we applied the forward cap covering the exercise period as control variate with sampling at the exercise time. Bermduan exercise premium and the duality gap are in basis points.

Figure 4: Duality Gaps for single-factor RS_P and multi-factor LSM exercise strategies



This figure demonstrates that the duality gaps from the single-factor RS_P exercise strategy are at least as low as the multi-factor LSM, when we use the same number of paths. On the first axis we have the set of Bermudan swaptions, while the second-axis contains the duality gaps measured in basis points. 95% confindence intervals have been included as error bars on the data series.

On top of that we have seen that the conditional present value losses based on 5,000 paths where not significantly different from zero. Even if the expected losses were positive, it seems reasonable that we also must worry about the variance of these conditional losses. If the variance is much larger than the expected loss, the holders of Bermudan swaptions could just as well get lucky. Consider, for example, a dealer, that repeatedly writes an ATM 10nc1 Bermudan swaption to a "fool", who is just following a single-factor exercise strategy in a multi-factor world. Our results show that with these odds, not even 5,000 deals are enough for the Law of Large Numbers to have locked in an almost sure profit. Hence, we claim that expected losses from following single-factor exercise strategies are also economically irrelevant.

Finally, we stress that these findings do not prove that single-factor models are able to neither hedge nor value Bermudan swaptions properly. In our opinion the ultimate test of the performance of single-factor model in the Bermudan swaptions market is their hedging performance. As already mentioned, the literature on pricing and hedging of interest rate derivatives has been growing rapidly. Hot topics of relevance in the Bermudan swaptions market, is the concept of unspanned stochastic volatility (USV) recently introduced in Collin-Dufresne & Goldstein (2002), who find that straddles in the cap and floor markets cannot be hedged using bonds alone. On the other hand, Fan, Gupta & Ritchken (2002) find that the swaptions market is well integrated with Libor swap rates and find no evidence of USV. Another research topic with relevance for the valuation of Bermudan swaption is the modelling of stochastic volatility and the importance of non-monotonic skews in the implied volatilities. A stochastic factor driving volatilities is likely to have a much larger effect on Bermudan swaption values than just another yield curve factor (see e.g. Jensen & Svenstrup (2002) for some preliminary tests). Recent papers discussing the implementation and calibration of Libor market models with stochastic volatility include Andersen & Brotherton-Ratcliffe (2001), Joshi & Rebonato (2001).

9 Conclusion

We find that following exercise strategies from calibrated single-factor models in a multi-factor world does not necessarily lead to significant losses as claimed in the literature. Neither are there any indications that the conditional present value losses introduced in Longstaff et al. (2001) are important sources of risk for Bermudan swaptions. Our findings show that the losses reported in Longstaff et al. (2001) cannot be ascribed to the number of factors in the model determining the exercise strategy.

Interestingly we also find that the LSM approach outperforms the LAM approach when valuing Bermudan swaptions with long exercise periods. In fact, even the exercise strategy from the worst performing single-factor model performs better.

References

Andersen, L. (2000), 'A simple approach to the pricing of bermudan swaptions in the multifactor LIBOR market model', *Journal of Computational Finance* **3**(2), 5–32.

Andersen, L. & Andreasen, J. (2000), 'Volatility skews and extensions of the libor market model', *Applied Mathematical Finance* **7**, 1–32.

Andersen, L. & Andreasen, J. (2001), 'Factor dependence of bermudan swaptions: Fact or fiction?', *Journal of Financial Economics* **62**, 3–37.

Andersen, L. & Broadie, M. (2001), 'A primal-dual simulation algorithm for pricing multi-dimensional american options', *Working Paper: General Re Securities and Columbia University* .

Andersen, L. & Brotherton-Ratcliffe, R. (2001), 'Extended libor market models with stochastic volatility', *Working Paper: Gen Re Securities* .

Andreasen, J. (2000), 'Turbo charging the cheyette model', *Working Paper, General Re Securities* .

Bjerregaard Pedersen, M. (1999), 'Bermudan swaptions in the LIBOR market model', *Financial Research Department, SimCorp A/S* .

Black, F., Derman, E. & Toy, W. (1990), 'A one-factor model of interest rates and its application to treasury bond options', *Financial Analyst Journal* pp. 33–39.

Boyle, P., Brodie, M. & Glasserman, P. (1997), 'Monte carlo methods for security pricing', *Journal of Economic Dynamics and Control* **21**, 1267–1321.

Brace, A., D. G. & Musiela, M. (1997), 'The market model of interest rate dynamics', *Mathematical Finance* **7**, 127–154.

Bühler, W., Uhrig-Homburg, M., Walter, U. & Weber, T. (1999), 'An empirical comparison of forward-rate and spot-rate models for valuing interest-rate options', *The Journal of Finance* **LIV**(1), 269–305.

Carverhill, A. (1995), 'A note on the models of hull and white for pricing options on the term structure', *The Journal of Fixed Income* pp. 89–96.

Collin-Dufresne, P. & Goldstein, R. S. (2002), 'Do bonds span the fixed income markets? theory and evidence for unspanned stochastic volatility.', *Journal of Finance* **LVII**(4), 1685–1730.

Driessen, J., Klaassen, P. & Melenberg, B. (2000), 'The performance of multi-factor term structure models for pricing and hedging caps and swaptions', *Working Paper, Tilburg University - Center for Economic Research* .

Duffie, D. (1996), *Dynamic Asset Pricing*, Princeton University Press.

Fan, R., Gupta, A. & Ritchken, P. (2002), 'Hedging in the possible presence of unspanned stochastic volatility: Evidence from swaption markets', *Workin Paper Case Western Reserve University, WSOM* .

Green, T. C. & Figlewski, S. (1999), 'Market risk and model risk for a financial institution writing options', *The Journal of Finance* **LIV**(4), 1465–1499.

Gupta, A. & Subrahmanyam, M. G. (2002), 'Pricing and hedging interest rate options: Evidence from cap-floor markets', *Working Paper: Case Western Reserve University and Stern School of Business* .

Haugh, M. B. & Kogan, L. (2001), 'Pricing american options: A duality approach', *Working Paper: Massachusetts Institute of Technology and University of Pennsylvania* .

Heath, D., Jarrow, R. & Morton, A. (1992), 'Bond pricing and the term structure of interest rates: A new methodology', *Econometrica* **60**, 77–105.

Hull, J. & Suo, W. (2001), 'A methodology for assessing model risk and its application to the implied volatility function model', *Working Paper, Joseph L. Rotman School of Management, University of Toronto* .

Hull, J. & White, A. (1995), '"a note on the models of hull and white for pricing options on the term structure": Response', *The Journal of Fixed Income* pp. 97–102.

Hull, J. & White, A. (1999), 'Forward rate volatilities, swap rate volatilites, and the implementation of the libor market model', *Working Paper, Joseph L. Rotman School of Management, University of Toronto* .

James, J. & Webber, N. (2000), *Interest Rate Modelling*, John Wiley and Sons, Ltd.

Jamshidian, F. (1991), 'Forward induction and construction of yield curve diffusion models', *Journal of Fixed Income* **1**, 62–74.

Jamshidian, F. (1997), 'Libor and swap market models and measures', *Finance and Stochastics* **1**, 293–330.

Jensen, M. S. & Svenstrup, M. (2002), 'Efficient control variates and strategies for bermudan swaptions in a libor market model', *Working Paper, University of Aarhus and Aarhus School of Business* .

Joshi, M. & Rebonato, R. (2001), 'A stochastic-volatility, displaced-diffusion extension of the LIBOR market model', *QUARC working paper* .

Li, A., Ritchken, P. & Sankarasubramanian, L. (1995), 'Lattice models for pricing american interest rate claims', *The Journal of Finance* **L**(2), 719–737.

Littermann, R. & Scheinkman, J. (1991), 'Common factors affecting bond returns', *Journal of Fixed Income* pp. 54–61.

Litzenberger, R. H. (1992), 'Swaps: Plain and fanciful', *Journal of Finance* **47**(3), 831–850.

Longstaff, F. A., Santa-Clara, P. & Schwartz, E. S. (2001), 'Throwing away a billion dollars: The cost of suboptimal exercise strategies in the swaptions market', *Journal of Financial Economics* **62**(62), 39–66.

Longstaff, F. A. & Schwartz, E. S. (2001), 'Valuing american options by simulation: A simple least-squares approach', *The Review of Financial Studies* **14**(1), 113–147.

Miltersen, K., Sandmann, K. & Sondermann, D. (1997), 'Closed form solutions for term structure derivatives with log-normal interest rates', *Journal of Finance* **52**, 409–430.

Press, W., Flannery, B., Teukolsky, S. & Vetterling, W. (1989), *Numerical Recipes in Pascal*, Cambridge University Press.

Rasmussen, N. S. (2002), 'Efficient control variates for monte-carlo valuation of american options', *Working Paper, The Aarhus School of Business* .

Rebonato, R. (1996), *Interest-Rate Option Models*, second edn, John Wiley and Sons.

Rebonato, R. (2000), 'Accurate and optimal calibration to co-terminal european swaptions in a FRA-based BGM framework', *Quarc working paper* .

Rebonato, R. (n.d.*a*), 'Theory and practice of model risk management', *working paper, Quantitative Research Centre (QUARC) of the Royal Bank of Scotland* .

Rebonato, R. (n.d.*b*), *Volatility and Correlation*, John Wiley and Sons, LTD.

Ritchken, P. & Sankarasubramanian, L. (1995), 'Volatility structures of forward rates and the dynamics of the term structure', *Mathematical Finance* **5**, 55–72.

Rithcken, P. & Chuang, I. (1999), 'Interest rate option pricing with volatility humps', *Review of Derivatives Research* **3**, 237–262.

Santa-Clara, P. & Sornette, D. (2001), 'The dynamics of the forward interest rate curve with stochastic string shocks', *The Review of Financial Studies* **14**(1), 149–185.

A Appendix

A.1 EUR Factor Loadings

Figure 5: Factor loadings in the benchmark model

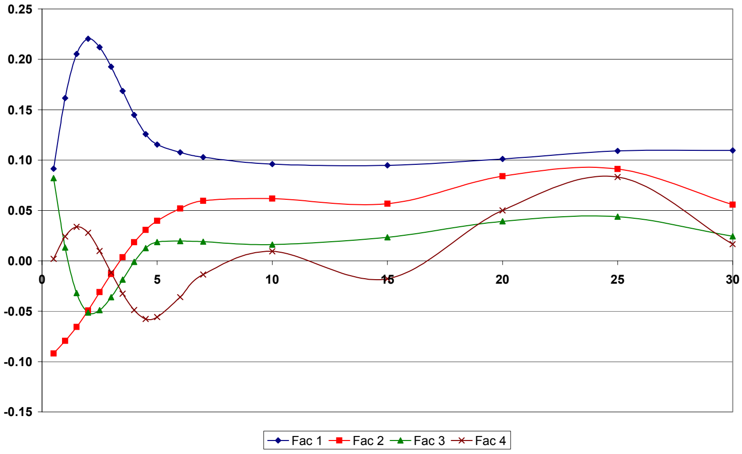


Illustration of the factor loadings for the 4-factor log-normal Libor market model used as benchmark.

Table 7: Factor loadings in the benchmark model

Tenor	0.50	1.00	1.50	2.00	2.50	3.00	3.50	4.00	4.50
1	0.091	0.162	0.205	0.220	0.212	0.193	0.169	0.145	0.126
2	-0.092	-0.079	-0.066	-0.049	-0.031	-0.013	0.004	0.018	0.031
3	0.082	0.013	-0.032	-0.051	-0.049	-0.036	-0.019	-0.001	0.013
4	0.002	0.024	0.034	0.028	0.010	-0.012	-0.033	-0.049	-0.058
Tenor	5.00	6.00	7.00	10.00	15.00	20.00	25.00	30.00	
1	0.115	0.108	0.103	0.096	0.095	0.101	0.109	0.110	
2	0.040	0.052	0.060	0.062	0.057	0.084	0.091	0.056	
3	0.019	0.020	0.019	0.016	0.023	0.039	0.044	0.024	
4	-0.056	-0.036	-0.014	0.009	-0.018	0.050	0.083	0.017	

Factor loadings defining the log-normal benchmark Libor market model.

A.2 Term Correlations

The term correlation ρ^* from time t to T between two variables X_k and X_j with instantaneous variances of $\sigma_i(s)$,$i = k, j$ and instantaneous correlation $\rho_{kj}(s)$ is defined as (see e.g. Rebonato (n.d.b))

$$\rho^*_{kj}(t,T) = \frac{\int_t^T \sigma_k(s)\,\sigma_j(s)\,\rho_{kj}(s)ds}{\sqrt{\int_t^T \sigma_k^2(s)\,ds \int_t^T \sigma_j^2(s)\,ds}}.$$

A.3 Vasicek Term-Correlation

Consider the Vasicek model where the short rate dynamics is given by the SDE

$$dr(t) = \kappa(\theta - r(t))\,dt + \sigma dW(t),$$

which have a closed form solution, $t < u$,

$$r(u) = \theta + e^{-\kappa(u-t)}(r(t) - \theta) + \sigma \int_t^u e^{-\kappa(u-a)} dW(a).$$

The variance of

$$\begin{aligned} Var(r(t)) &= Var\left(\sigma \int_t^u e^{-\kappa(u-a)} dW(a)\right) = \sigma^2 \int_t^u e^{-2\kappa(u-a)} dW(a) \\ &= \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right). \end{aligned}$$

The time zero term-covariance is easily seen to be $t < u$,

$$\begin{aligned} Cov(r(t), r(u)) &= Cov\left(r(t), e^{-\kappa(u-t)} r(t) + \sigma \int_t^u e^{-\kappa(u-a)} dW(a)\right) \\ &= e^{-\kappa(u-t)} Var(r(t)). \end{aligned}$$

And from this the term-correlation function is derived

$$\begin{aligned} corr(r(t), r(u)) &= \frac{Cov(r(t), r(u))}{\sqrt{Var(r(t))\,Var(r(u))}} = \frac{e^{-\kappa(u-t)} Var(r(t))}{\sqrt{Var(r(t))\,Var(r(u))}} \\ &= e^{-\kappa(u-t)} \sqrt{\frac{Var(r(t))}{Var(r(u))}} = e^{-\kappa(u-t)} \sqrt{\frac{(1 - e^{-2\kappa t})}{(1 - e^{-2\kappa u})}} \\ &= \sqrt{\frac{(1 - e^{2\kappa t})}{(1 - e^{2\kappa u})}}. \end{aligned}$$

A.4 Conditional Losses

Table 8: Comparison of single-factor BDT_V and four-factor Libor market model exercise strategies

Swaption			Probability of Exercise %		Single-factor Exercises						
						Same		Early		Late	
t_s	t_e	θ	LSM	BDT_V	Early	time	Late	Loss	Std	Loss	Std
1	10	4%	90.3	88.0	0.1	76.2	23.7	-12	(41)	15	(10)
1	10	5%	67.6	64.4	0.1	78.9	21.0	228	(67)	15	(10)
1	10	6%	46.4	43.7	0.1	86.5	13.4	296	(155)	29	(10)
3	10	4%	85.1	83.5	0.1	88.2	11.7	-13	(39)	11	(9)
3	10	5%	65.2	62.9	0.0	85.5	14.4	85	(21)	-3	(9)
3	10	6%	45.3	42.8	0.0	88.7	11.3	58	-	21	(10)
6	10	4%	77.0	75.7	0.3	94.5	5.2	-61	(30)	9	(5)
6	10	5%	59.2	57.3	0.0	92.4	7.6	0	-	14	(6)
6	10	6%	40.7	38.5	0.0	93.1	6.9	0	-	16	(7)
1	15	4%	89.7	87.5	0.4	70.6	29.0	72	(45)	33	(13)
1	15	5%	70.2	67.0	0.4	74.6	25.0	-3	(72)	2	(12)
1	15	6%	52.4	49.5	0.1	82.0	17.9	-190	(128)	23	(12)

This table reports summary statistics for the single-factor exercise strategy and the Least Square Monte Carlo exercise strategy in the multi-factor Libor market model. Probability of exercise represents the total percentage of paths for which the swaption is exercised. Also the percentage of paths where the single-factor models signals exercise earlier, at the same time, or later than the multi-factor model are included. The present value losses of exercising when the single-factor model implies it is optimal at an earlier time than the four-factor model is the difference between the immediate value of exercise and the present value of cash flows generated by following the multi-factor strategy, averaged over all paths where the single-factor implies exercise earlier than. Similarly, for the present value loss of exercising when the single-factor model implies that exercise is optimal at a later time than the four-factor model. All costs are expressed in basis points. Values are based on 5,000 simulated paths of the term structure.

Table 9: Comparison of single-factor BDT_P and four-factor Libor market model exercise strategies

Swaption			Probability of Exercise %		Single-factor Exercises						
					Same			Early		Late	
t_s	t_e	θ	LSM	BDT_P	Early	time	Late	Loss	Std	Loss	Std
1	10	4%	89.4	88.1	0.5	82.3	17.3	-3	(43)	38	(12)
1	10	5%	68.6	66.9	0.8	83.5	15.6	0	(11)	31	(13)
1	10	6%	47.2	46.1	0.5	93.0	6.5	-58	(11)	31	(16)
3	10	4%	85.0	84.5	0.8	93.5	5.7	25	(8)	13	(15)
3	10	5%	65.9	65.0	0.6	92.7	6.6	-53	(10)	21	(14)
3	10	6%	46.0	45.2	0.4	93.7	5.9	-35	(11)	-25	(15)
6	10	4%	77.5	77.0	0.8	97.2	2.0	-27	(6)	2	(10)
6	10	5%	57.4	56.7	0.6	96.4	3.0	3	(6)	-5	(11)
6	10	6%	40.5	40.0	0.9	97.1	2.1	-21	(8)	22	(12)
1	15	4%	89.5	87.5	1.2	72.4	26.4	-66	(14)	20	(13)
1	15	5%	68.1	66.6	0.9	80.3	18.8	-111	(13)	16	(14)
1	15	6%	52.5	51.0	0.6	87.3	12.1	4	(12)	11	(15)

This table reports summary statistics for the single-factor exercise strategy and the Least Square Monte Carlo exercise strategy in the multi-factor Libor market model. Probability of exercise represents the total percentage of paths for which the swaption is exercised. Also the percentage of paths where the single-factor models signals exercise earlier, at the same time, or later than the multi-factor model are included. The present value losses of exercising when the single-factor model implies it is optimal at an earlier time than the four-factor model is the difference between the immediate value of exercise and the present value of cash flows generated by following the multi-factor strategy, averaged over all paths where the single-factor implies exercise earlier than. Similarly, for the present value loss of exercising when the single-factor model implies that exercise is optimal at a later time than the four-factor model. All costs are expressed in basis points. Values are based on 5,000 simulated paths of the term structure.

Table 10: Comparison of single-factor RS_P and four-factor Libor market model exercise strategies

Swaption			Probability of Exercise %		Single-factor Exercises						
					Same			Early		Late	
t_s	t_e	θ	LSM	RS_P	Early	time	Late	Loss	Std	Loss	Std
1	10	4%	90.3	90.6	9.8	87.1	3.1	-8	(14)	9	(22)
1	10	5%	67.2	66.5	6.9	87.1	6.0	-8	(16)	1	(17)
1	10	6%	46.0	44.9	2.7	92.7	4.6	9	(20)	47	(20)
3	10	4%	84.5	84.8	9.8	88.0	2.3	4	(11)	2	(16)
3	10	5%	65.8	65.4	5.2	91.7	3.1	-22	(16)	-5	(16)
3	10	6%	47.3	46.1	3.0	92.3	4.7	20	(18)	13	(16)
6	10	4%	77.9	77.5	6.3	91.6	2.1	4	(8)	6	(6)
6	10	5%	59.2	58.8	3.1	94.7	2.2	-10	(13)	9	(7)
6	10	6%	42.2	41.8	1.8	96.4	1.7	-9	(17)	3	(9)
1	15	4%	89.4	89.8	14.9	80.5	4.6	-13	(16)	-20	(25)
1	15	5%	70.9	70.0	9.1	82.7	8.2	-13	(18)	10	(20)
1	15	6%	51.5	50.0	4.3	87.8	7.9	-2	(22)	23	(18)

This table reports summary statistics for the single-factor exercise strategy and the Least Square Monte Carlo exercise strategy in the multi-factor Libor market model. Probability of exercise represents the total percentage of paths for which the swaption is exercised. Also the percentage of paths where the single-factor models signals exercise earlier, at the same time, or later than the multi-factor model are included. The present value losses of exercising when the single-factor model implies it is optimal at an earlier time than the four-factor model is the difference between the immediate value of exercise and the present value of cash flows generated by following the multi-factor strategy, averaged over all paths where the single-factor implies exercise earlier than. Similarly, for the present value loss of exercising when the single-factor model implies that exercise is optimal at a later time than the four-factor model. All costs are expressed in basis points. Values are based on 5,000 simulated paths of the term structure.

Table 11: Comparison of single-factor RS_P and four-factor Libor market model exercise strategies

Swaption			Probability of Exercise %		Single-factor Exercises						
						Same		Early		Late	
t_s	t_e	θ	LSM	RS_V	Early	time	Late	Loss	Std	Loss	Std
1	10	4%	90.2	89.9	12.5	83.3	4.3	-7	(11)	14	(21)
1	10	5%	67.4	67.2	12.4	81.1	6.5	-29	(11)	-20	(15)
1	10	6%	46.0	44.8	7.3	87.1	5.6	1	(11)	44	(16)
3	10	4%	84.4	85.1	15.7	82.3	2.1	-1	(8)	-14	(18)
3	10	5%	64.0	63.7	11.4	84.7	3.9	4	(10)	27	(13)
3	10	6%	45.8	44.8	7.1	88.4	4.5	6	(11)	14	(15)
6	10	4%	77.4	77.7	12.0	86.1	1.9	-1	(6)	-4	(8)
6	10	5%	57.8	57.5	8.5	89.5	2.0	-1	(6)	5	(8)
6	10	6%	42.4	42.0	6.1	91.9	1.9	-11	(8)	10	(7)
1	15	4%	89.4	89.3	15.7	78.6	5.7	4	(14)	48	(24)
1	15	5%	70.1	68.9	11.7	78.0	10.3	13	(13)	29	(18)
1	15	6%	52.8	50.9	7.3	83.7	9.0	4	(12)	46	(19)

This table reports summary statistics for the single-factor exercise strategy and the Least Square Monte Carlo exercise strategy in the multi-factor Libor market model. Probability of exercise represents the total percentage of paths for which the swaption is exercised. Also the percentage of paths where the single-factor models signals exercise earlier, at the same time, or later than the multi-factor model are included. The present value losses of exercising when the single-factor model implies it is optimal at an earlier time than the four-factor model is the difference between the immediate value of exercise and the present value of cash flows generated by following the multi-factor strategy, averaged over all paths where the single-factor implies exercise earlier than. Similarly, for the present value loss of exercising when the single-factor model implies that exercise is optimal at a later time than the four-factor model. All costs are expressed in basis points. Values are based on 5,000 simulated paths of the term structure.

Table 12: Comparison of single-factor AN_P and four-factor Libor market model exercise strategies

Swaption			Probability of Exercise %		Single-factor Exercises						
						Same		Early		Late	
t_s	t_e	θ	LSM	AN_P	Early	time	Late	Loss	Std	Loss	Std
1	10	4%	90.0	89.7	2.2	90.7	7.1	21	(23)	21	(18)
1	10	5%	68.5	68.1	3.0	91.7	5.3	9	(19)	40	(19)
1	10	6%	48.1	47.9	1.6	95.3	3.1	-10	(20)	-1	(20)
3	10	4%	84.2	84.4	3.1	94.0	2.9	-6	(15)	23	(21)
3	10	5%	65.2	64.9	2.6	94.0	3.5	-22	(18)	2	(17)
3	10	6%	46.7	46.3	2.0	95.0	2.9	-1	(17)	-10	(22)
6	10	4%	76.6	76.4	4.8	93.9	1.3	-4	(8)	-2	(10)
6	10	5%	58.7	58.6	2.1	96.9	1.1	-5	(13)	1	(8)
6	10	6%	42.3	41.9	1.3	97.5	1.2	-10	(14)	27	(20)
1	15	4%	90.1	90.1	7.3	87.7	4.9	-6	(20)	11	(26)
1	15	5%	70.6	70.5	6.0	88.7	5.2	-32	(20)	4	(21)
1	15	6%	51.7	51.7	2.9	92.6	4.5	-57	(24)	-13	(21)

This table reports summary statistics for the single-factor exercise strategy and the Least Square Monte Carlo exercise strategy in the multi-factor Libor market model. Probability of exercise represents the total percentage of paths for which the swaption is exercised. Also the percentage of paths where the single-factor models signals exercise earlier, at the same time, or later than the multi-factor model are included. The present value losses of exercising when the single-factor model implies it is optimal at an earlier time than the four-factor model is the difference between the immediate value of exercise and the present value of cash flows generated by following the multi-factor strategy, averaged over all paths where the single-factor implies exercise earlier than. Similarly, for the present value loss of exercising when the single-factor model implies that exercise is optimal at a later time than the four-factor model. All costs are expressed in basis points. Values are based on 5,000 simulated paths of the term structure.

Table 13: Comparison of single-factor AN_V and four-factor Libor market model exercise strategies

Swaption			Probability of Exercise %		Single-factor Exercises						
						Same		Early		Late	
t_s	t_e	θ	LSM	AN_V	Early	time	Late	Loss	Std	Loss	Std
1	10	4%	89.9	89.5	4.3	88.3	7.4	-19	(14)	30	(18)
1	10	5%	68.6	68.0	5.3	90.2	4.5	5	(11)	3	(21)
1	10	6%	46.1	45.6	3.9	93.8	2.4	17	(11)	24	(23)
3	10	4%	84.7	84.5	6.6	91.9	1.6	-1	(8)	-33	(22)
3	10	5%	63.9	63.8	5.8	91.8	2.4	-2	(10)	13	(18)
3	10	6%	46.7	46.1	4.0	93.1	2.9	-15	(11)	7	(20)
6	10	4%	78.0	78.1	9.5	89.0	1.6	-10	(6)	20	(10)
6	10	5%	58.4	58.0	5.5	92.9	1.7	-11	(6)	5	(9)
6	10	6%	41.5	41.3	4.2	94.6	1.2	-4	(8)	-17	(11)
1	15	4%	89.8	89.9	5.6	90.0	4.3	-19	(14)	4	(26)
1	15	5%	70.5	70.0	6.9	88.4	4.7	-23	(13)	12	(24)
1	15	6%	53.0	52.2	4.0	91.1	4.9	-2	(12)	10	(24)

This table reports summary statistics for the single-factor exercise strategy and the Least Square Monte Carlo exercise strategy in the multi-factor Libor market model. Probability of exercise represents the total percentage of paths for which the swaption is exercised. Also the percentage of paths where the single-factor models signals exercise earlier, at the same time, or later than the multi-factor model are included. The present value losses of exercising when the single-factor model implies it is optimal at an earlier time than the four-factor model is the difference between the immediate value of exercise and the present value of cash flows generated by following the multi-factor strategy, averaged over all paths where the single-factor implies exercise earlier than. Similarly, for the present value loss of exercising when the single-factor model implies that exercise is optimal at a later time than the four-factor model. All costs are expressed in basis points. Values are based on 5,000 simulated paths of the term structure.